



University of Connecticut

Department of Economics Working Paper Series

**Existence and Uniqueness of Equilibrium in Nonoptimal Un-
bounded Infinite Horizon Economies**

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Working Paper 2001-02

April 2001

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Abstract

In applied work in macroeconomics and finance, nonoptimal infinite horizon economies are often studied in the the state space is unbounded. Important examples of such economies are single vector growth models with production externalities, valued fiat money, monopolistic competition, and/or distortionary government taxation. Although sufficient conditions for existence and uniqueness of Markovian equilibrium are well known for the compact state space case, no similar sufficient conditions exist for unbounded growth. This paper provides such a set of sufficient conditions, and also present a computational algorithm that will prove asymptotically consistent when computing Markovian equilibrium.

Journal of Economic Literature Classification: C62, D58, D62

1 INTRODUCTION

This paper establishes existence and uniqueness results for a broad class of dynamic equilibrium nonoptimal single sector stochastic unbounded growth models often used in applied macroeconomics and public finance. The primitive data describing the class of models under consideration include economies with a diverse set of potential equilibrium distortions such as distortionary taxes, valued fiat money, monopolistic competition, and various types of production externalities. While there is vast literature on unbounded and endogenous growth, there are no known results establishing sufficient conditions under which there exist Markovian equilibria for such models, let alone sufficient conditions under which the equilibrium is unique.¹

The methods developed in this paper are constructive, and therefore allow us to discuss issues associated with computation as well as characterization. Further, the methods are not topological, but rather built around the limit of a particular trajectory of a nonlinear operator constructed from an equilibrium version of the household's Euler equation. This operator is defined on a domain that possesses a desirable chain completeness property for particular trajectories, and is shown to have a fixed point. In this sense, our methodology is related to the Euler equation approaches discussed in the important recent work of both Coleman ([10][11]) and Greenwood and Huffman [15], and which establish the equilibrium as a fixed point of a nonlinear mapping.

Since the seminal work of Arrow and Debreu, fixed point theorems have been at the core of general equilibrium analysis. Early work on existence of competitive equilibrium appealed to topological constructions such as Brouwer's fixed point theorem, a theorem asserting that a single-valued continuous mapping from a compact convex subset of a vector space into itself has a fixed point. In more general situations, Kakutani's fixed point theorem, Schauder's fixed point theorem, or the Fan-Glicksberg fixed point theorem are often required. All of these theorems are topological in nature, and essentially extend the result of Brouwer to the case of correspondences and infinite dimensional spaces. In the context of a recursive dynamic monetary economy, Lucas and Stokey [17] for example, apply Schauder's fixed point theorem to establish that a nonlinear operator that maps a non-empty, closed, bounded and convex subset of continuous functions $C(X)$ defined on a compact subset X into itself has a fixed point if it is continuous and if the underlying subset is an equicontinuous set of functions. In the work of Jovanovic [16], Bernhardt and Bergin [7], and most

¹The papers that study problems most closely related to our work are Coleman ([10], [11]) and Greenwood and Huffman [15]. However, the results in those papers critically depend on production satisfying a particular uniform boundedness condition which implies compactness of the state space.

The methods in these papers are based upon topological versions Tarski's theorem (Tarski [23] and Dugundji and Granas [13]) which are not available for our environments as the underlying set of functions for which we pose the existence of equilibrium is not a countably chain complete lattice. Further, versions of Tarski's theorem for partially ordered Banach spaces (e.g., Amann [5]) are also not available, as our space is not a subset of an Banach space of continuous functions.

recently Chakrabarti [9], generalizations of Schauder's theorem for correspondences, the so-called Fan-Glicksberg class of fixed point theorems, are used to establish the existence of equilibrium for a class of large anonymous games and heterogeneous agent economies.

There are, however, some major impediments to applying these topological constructions to the class of unbounded growth models considered in this paper. First, while the theorems of Schauder and Fan-Glicksberg are existential, the operators we study often contain trivial fixed points that cannot be decentralized under a price system with strongly concave households. Ruling out trivial fixed points would therefore require constructing domains of functions that exclude from consideration such trivial elements, which make these theorems difficult to apply since finding such domains is often an intractable problem. Second, to apply these theorems, the state space X has to be compact, which is never the case for unbounded growth models. Additionally, proving continuity of a nonlinear operator in a particular topology when the state space is not compact is often very difficult.

This leads one to consider fixed point arguments that are not topological and more specific to the problem under consideration, i.e., that exploit some additional structure of the particular problem being studied. An important class of arguments that exploits the underlying completeness of the domain of the nonlinear operator is based on the Contraction Mapping Theorem. A commonly applied theorem in recursive equilibrium theory, the Contraction Mapping theorem asserts that an operator that is a contraction mapping of a complete metric space into itself has exactly one fixed point. No topological considerations for this operator are required; rather a particular type of completeness (namely Cauchy completeness) is the structure needed.

Another interesting application of a non topological fixed point theorem is the case of bounded growth models with equilibrium distortions: Coleman [10] pioneered an application of a version of Tarski's fixed point theorem (by Dugundji and Granas) to demonstrate existence of equilibrium in an infinite horizon stochastic framework with an income tax. Tarski's fixed point theorem in its most general form (see Aliprantis and Border [4]), establishes that a lattice monotone operator from a countably chain complete lattice into itself has a fixed point. Further, if two elements, l and u , exist such that $A(l) \geq l$ and $A(u) \leq u$, a version of Tarski's theorem requiring continuity of A also provides a method for computing for the minimal and maximal fixed points (see Dugundji and Granas [13]).

Coleman's method thus seems promising for our problem. However, for the class of models studied in Coleman [10], a standard restriction on the production function insures that the state space X is compact (in addition to being convex and closed). This fundamental property of the model allows the author to establish, through the Arzela-Ascoli theorem, that a set of equicontinuous functions (endowed with the sup norm) defined on X is a compact subset of a lattice of continuous functions. Compactness is then a sufficient condition to

guarantee that this subset is a countably chain complete lattice.² Coleman then constructs a monotone continuous operator from this complete lattice set into itself, and an application of a particular version of Tarski's fixed point theorem generates an algorithm that converges to the fixed point, shown to be unique and strictly positive. Unfortunately, one of the key elements of chain complete sets is boundedness, which is not available in the case of distorted unbounded growth models when X is not compact. The strategy in Coleman [10] thus cannot be directly applied to our problem.

It is interesting to note that, just as Brouwer's result can be extended to upper semi-continuous correspondences (Kakutani), Tarski's theorem can be extended to ascending sublattice valued correspondences (e.g. Zhou [24]), thus providing a new set of tools for arguments for proofs of existence and for constructions of characterizations of how all equilibria vary in a parameter (e.g., a measure of distortionary taxes). For the case of bounded growth, the recent paper by Mirman, Morand, and Reffett [18] extends the results of Coleman [10] to a much broader class of distorted environments via an application of the main theorem of Zhou [24]. The authors show how "ordered" increases in state dependent income taxes affect the equilibrium manifold for a given class of bounded growth models. Unfortunately, the compactness assumption of the state space again implies that none of those results can be trivially extended to unbounded growth settings either.

In principle, assuming that a complete lattice of functions suitable for a version of Tarski's fixed point theorem can be produced, it would appear that a similar argument to Coleman [10] could be constructed to address existence and uniqueness in unbounded growth frameworks. Unfortunately, it appears to be the case that, generally, subsets of equilibrium functions which can be characterized easily and that form a complete lattice have the property that the state space X is appropriately shown to be compact.³ An alternative strategy is to apply a fixed point theorem that combines continuity of an operator mapping an ordered Banach space into itself with order preserving monotonicity to deliver, under some conditions, a minimal and maximal fixed point (see Amann, [5]). However, interesting Banach spaces of functions are hard to come by for our environments, because they are vector spaces, which, by definition, rule out any explicit bound imposed on the functions (such as resource or budget constraints).

In this paper, rather than searching for conditions on the combined structure of the space and of the operator that are sufficient for the existence of a fixed point, we propose a more practical and direct strategy that exploits the particulars of our environment. The next section of the paper presents the

²Birkhoff [8] shows that if a set is compact in its interval topology, it is equivalent to a complete lattice. It is easily shown that in the case studied by Coleman, since the equilibrium sits in a closed, pointwise compact, equicontinuous subset of continuous consumptions functions on a compact state space in the uniform topology, it is therefore a compact set in the interval topology. Consequently, it is a complete lattice, and therefore countably chain complete.

³One point compactification of R^+ does not help, since it allows for the added point to become the equilibrium state.

class of environments studied. In Section 3 we construct a set of fixed points of an operator A , and the strategy requires adapting a version of the fixed point theorems in Amann [5] for a set of hypotheses verified by our operator A . This establishes the existence of Markovian equilibrium. In Section 4, we show that all the fixed points of the mapping A are fixed points of another mapping, denoted \hat{A} , and that \hat{A} that has at most one interior fixed point. This forms the basis of our uniqueness argument. Section 5 concludes.

2 THE MODEL

Time is discrete and indexed by $t \in T = \{0, 1, 2, \dots\}$, and there is a continuum of infinitely-lived and identical household/firm agents. The aggregate state variables for this economy consist of endogenous and exogenous variables and are denoted by the vector S . Uncertainty comes in the form of a finite state first order Markov process denoted by $z_t \in \mathbb{Z}$ stationary transition probabilities $\pi(z, z')$.⁴ Let the set $\mathbb{K} \subset \mathbb{R}_+$ contain all the feasible values for the aggregate endogenous state variable K , and define the product space $\mathbb{S} : \mathbb{K} \times \mathbb{Z}$. Since the household will also enter a period with an individual level of the endogenous state variable k , we denote the state of a household by the vector $s = (k, S)$ with $s \in \mathbb{K} \times \mathbb{S}$. We assume that the class of equilibrium distortions are consistent with the representative agent facing a set of feasible constraints summarized by a correspondence $\Omega(k, k', S) \subset \mathbb{K} \times \mathbb{K} \times \mathbb{S}$ in which k' is next period value of the variable k . While more specific details will be provided below, for now we can think of Ω as simply the graph of the non-empty, continuous, convex and compact valued feasible correspondence for the household $\Gamma(s) : \mathbb{K} \times \mathbb{S} \rightarrow \mathbb{K}$.

Each household assumes that the aggregate endogenous state variable evolves according to a continuous function $K' = h(K, z)$ and owns an identical production technology which exhibits constant returns to scale in private inputs for producing output goods. Production may also depend on the equilibrium level of inputs, and by allowing the technologies to be altered by per capita aggregates we therefore include the case of production externalities. Production takes place in the context of perfectly competitive markets for both the output good and the factors of production.

2.1 Assumptions

Each period, households are endowed with a unit of time which they supply inelastically to competitive firms. With the capital-labor ratio denoted by k ,

⁴We use a finite state space for the exogenous shocks just to simplify the exposition. Amending the discussion to note concerns with the existence of increasing measurable solutions and increasing measurable selections (i.e., the sup and the inf of the optimal policy correspondence), all the results of the paper follow the present constructions in this more general setting. See Hopenhaym and Prescott [?] for a discussion of how to handle shock processes with more general state spaces, and what additional restrictions this case places on the transition processes for the shocks.

and the per-capita counterpart of this measurement by K , we assume that the production possibilities are represented by a function $f(k, K, z)$.

Household inelastically supply their endowment of labor to perfectly competitive firms. As a result, a household's income before taxes and transfers is exactly:⁵

$$f(K, z) + (k - K)f_1(K, z)$$

where equilibrium has been imposed on the firms problem, namely $f(K, z) = f(K, K, z)$. The government taxes all income at the rate $t_1(K, z)$ and transfers the lump sum amount $t_2(K, z)$ to each household. In period t a household must decide on an amount c to consume, and the capital-labor ratio carried over to the next period is thus:

$$k' = (1 - t_1(K, z))[f(K, z) + (k - K)f_1(K, z)] + t_2(K, z) - c$$

We make the following assumptions on the primitives data.

Assumption 1. The production function $f(k, K, z)$ and the function governing taxes and transfers $t(K, z)$ are such that:

(i). *The production function $f : K \times K \times Z \rightarrow K$ is continuous and strictly increasing. Further, it is continuously differentiable in its first two arguments, and strictly concave in its first argument.*

(ii). *The production function satisfies $f(0, K, z) = 0$ and $\lim_{k \rightarrow 0} f_1(0, K, z) = \infty$ for all $(K, z) \in K \times Z$.*

(iii). *The tax and transfer policy functions t_1 and t_2 are continuous and increasing in both their arguments.*

(iv). *The quantity $(1 - t_1(K, z))f_1(K, K, z)$ is strictly decreasing in K .*

Aside from the lack of any boundedness condition on the production function f , these restrictions are standard (e.g., Coleman [10]). Given the nonexistence of continuous Markovian equilibrium results presented recently in the work of Santos [21] and Mirman, Morand, and Reffett [18], it would seem that assumptions (iii)-(iv) are necessary for the existence of continuous Markovian equilibrium.⁶

For each period and state, the preferences are represented by a period utility index $u(c_i)$, where $c_i \in \mathbb{K} \subset R_+$ is period i consumption. Letting $z^i = (z_1, \dots, z_i)$ denote the history of the shocks until period i , a household's lifetime preferences are defined over infinite sequences indexed by dates and histories $c = (c_{zi})$ and

⁵It is well known that economies with production externalities, monetary distortions, and monopolistic competition are observationally equivalent to models with taxes. Therefore although we notate the equilibrium distortions with taxes, we have in mind a much broader set of environments used in applied work.

⁶Santos [21] presents a counterexample which highlights the need for monotonicity of distorted returns in models such as ours. In particular, he produces a non-existence of continuous Markovian equilibrium result in which the key feature of the example is that the distorted return on capital is not monotone.

are given by:

$$U(\mathbf{c}) = E_o \left\{ \sum_{i=0}^{\infty} \beta^i u(c_i) \right\} \quad (1)$$

where the summation in (1) is with respect to the probability structure of future histories of the shocks z^i given the history of shocks, the transition matrix χ and the optimal plans up to a given date i .

Assumption 2. The period utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is bounded, continuously differentiable, strictly increasing, strictly concave, $0 < \beta < 1$, and $u'(0) = \infty$.

2.2 Value function and Equilibrium

The value function V associated with the household's problem of choosing an optimal consumption level satisfies the Bellman's equation:

$$V(k, K, z) = \sup_{c \in \Gamma(k, K, z)} \{u(c) + \beta E_z[V((1 - t_1(K, z))[f(k, K, z) \quad (2)$$

$$+ (k - K)f_1(k, K, z)] + t_2(K, z) - c, h(K, z), z')]\}$$

where the constraint set for the household's choice of consumption is the compact interval:

$$\Gamma(k, K, z) = [0, (1 - t_1(K, z))[f(k, K, z) + (k - K)f_1(k, K, z)] + t_2(K, z)]$$

Consider the complete metric space of bounded, continuous, real-valued functions $v : \mathbb{R}_+ \times \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$ equipped with the sup norm, and W the subset of functions that are weakly increasing and concave in their first argument. The following is a standard result in the literature (see for instance Stokey, Lucas and Prescott [22]). Note that v is generally not defined on a compact space.

Proposition 1. Under Assumptions 1-2, given any continuous aggregate investment function h and any transfer policy function t_2 there exists a unique v in W that satisfies Bellman's equation (2). Moreover, this v is strictly increasing and strictly concave in its first argument. The optimal policy $c(k, K, z)$ is single valued and continuous in its first argument.

Proof: Stokey, Lucas and Prescott [22].⁷ Note that the proof relies on applying the contraction mapping theorem for an operator defined on the Banach space $C(X)$ with the uniform topology, without requiring X to be compact.

It is important to also note that the optimal policy $c(k, K, z)$ is strictly positive, that is $c(k, K, z) > 0$ when $k > 0$ and $K > 0$. Suppose that this is not

⁷ Alvarez and Stokey [3] present an extension of this result in the case of unbounded utility and when the rate of growth of feasible sequences can be bounded.

the case, i.e., that there exists $(k, K) > 0$ such that $c(k, K, z) = 0$. Consider increasing consumption and decreasing investment by some amount $\varepsilon > 0$. The per unit increase in current utility is $[u(\varepsilon) - u(0)]/\varepsilon$, while the per unit decrease in expected future utility is $\beta E_t[v(k', K', z') - v(k' - \varepsilon, K', z')]/\varepsilon$. However:

$$\lim_{\varepsilon \rightarrow 0} [u(\varepsilon) - u(0)]/\varepsilon = u'(0) = \infty$$

and the utility gains can therefore be made arbitrarily large by choosing ε small enough, while the quantity loss are bounded since v is strictly increasing and concave in its first argument (and $k' > 0$ when $c(k, K, z) = 0$ and $K' = h(K, z) > 0$ as well). As a consequence, the policy of consuming nothing is not optimal.

We define an equilibrium as follows:

Definition: A stationary equilibrium consists of continuous functions (h, t_2) mapping $R_{++} * Z$ into R_{++} such that:

- (i). All tax revenues are lump-sum redistributed according to the transfer function $t_2 = t_1 f$.
- (ii). The aggregate investment function h is such that households choose to invest according to the same rule:

$$h(K, z) = f(K, K, z) - c(K, K, z)$$

Proposition 2. Under Assumptions 1-2 if (h, t_2) is an equilibrium with the associated policy function c and value function v , then $c(K, K, z)$ always lies in the nonempty interior of $\Gamma(K, K, z)$, and v is continuously differentiable in its first argument k when $k = K$ for all (K, z) .

Proof: Standard. See for instance Stokey, Lucas and Prescott.[22]

Consequently, denoting $c(K, z) = c(K, K, z)$, $H(K, z) = (1 - t_1(K, z))f_1(K, K, z)$ and $f(K, K, z) = F(K, z)$ for convenience, the optimal policy function necessarily satisfies the Euler equation:

$$u'(c(K, z)) = \beta E_z\{u'[c(F(K, z) - c(K, z), z')] * H(F(K, z) - c(K, z), z')\} \quad (3)$$

and an equilibrium consumption is a strictly positive solution $c(K, z) > 0$ to this Euler equation. Notice that the zero consumption is not an equilibrium.

3 EXISTENCE OF EQUILIBRIUM

As discussed previously, various versions of Tarski's fixed point theorem may be used to establish the existence of minimal and maximal fixed points for monotonic mappings in sets with special structures, and also to provide algorithms that converge to these particular fixed points. This is the type of methodology we will be using, and we first remind the reader of the basic Tarski's fixed point theorem.

Tarski's fixed point theorem (Tarski [23]). If f is an increasing mapping of a complete lattice into itself, then the set of fixed points is a nonempty complete lattice.

Abian and Brown [1] demonstrate that an increasing mapping f of a chain complete poset into itself has a fixed point if and only if it has an excessive or deficient point (resp. $f(a) \geq a$ and $f(b) \leq b$), and that in that event it has minimal and maximal fixed points.

Upon finding a particular excessive point, Coleman [10] applies an algorithm for an operator based on the equilibrium Euler equation for a bounded growth model that converges uniformly on a compact set of functions to the maximal fixed point (Dugundji and Granas [13]), and provides a set of sufficient conditions under which this maximal fixed point must be strictly positive. Greenwood and Huffman [15] provide a weaker set of sufficient conditions under which this same trajectory of the operator converges to a strictly positive fixed point. As emphasized by Coleman, the advantage of using a version of Tarski's fixed point theorem with an explicit algorithm is to be able to rule out the zero consumption as the maximal fixed point. However, one drawback of such methodology is that it seems to require the lattice continuity of the operator to establish that the limit of a particular sequence is the fixed point.

In the setup of this paper, the state space is not compact and therefore Coleman's set of equicontinuous functions is no longer compact subset of a Banach lattice of continuous functions. It therefore does not necessarily have the property that every chain has a supremum. However, we are not interested in the properties of every chain or every sequence: Rather, we study one particular sequence in an equicontinuous set of functions (but not compact) that can be shown to converge to a limit function in that equicontinuous set. These nice convergence characteristics are shown to depend on both the equicontinuity of the underlying space where we pose the fixed point problem -an order interval- as well as the monotonicity of the operator A (i.e., the order preserving property). Based on these properties, we formulate a fixed point theorem for at least one trajectory of the operator A , and its convergence to a stationary point requires neither a complete lattice nor chain completeness of the entire set.⁸

Our fixed point theorem is related to the results in Amann [5], who demonstrates that an increasing compact map from a non empty order interval into an ordered Banach space (or an ordered topological vector space) that maps one point down and another point up has a minimal and a maximal point. The proof of Amann's theorem requires establishing that a particular increasing sequence in an order interval has a limit point in that order interval. Our fixed point theorem exploits the fact that a sufficient condition for this to be true is that all sequences in the interval have a subsequence that converges in the interval, a property that we will show holds true in a set of functions we later define. When this condition holds, any increasing sequence converges to a unique limit

⁸There are fundamental fixed point theorems that give necessary and sufficient conditions for the existence of a fixed point of a mapping of a set S into itself where absolutely no structure is imposed on S . See Abian [2]

belonging to the set.

Theorem. Let $[\overline{y}, \widehat{y}]$ be an order interval of a lattice (E, \leq) , and $f : [\overline{y}, \widehat{y}] \rightarrow [\overline{y}, \widehat{y}]$ a continuous and monotone increasing function. If every sequence in $[\overline{y}, \widehat{y}]$ has a convergent subsequence that converges to an element in $[\overline{y}, \widehat{y}]$ then f has a minimal fixed point \overline{x} and a maximal fixed point \widehat{x} . Moreover, $\overline{x} = \lim_{k \rightarrow \infty} f^k(\overline{y})$ and $\widehat{x} = \lim_{k \rightarrow \infty} f^k(\widehat{y})$, and the sequences $\{f^k(\overline{y})\}_{k=0}^\infty$ and $\{f^k(\widehat{y})\}_{k=0}^\infty$ are increasing and decreasing, respectively.

Proof: Note first that, necessarily, $f(\overline{y}) \geq \overline{y}$ (and $\widehat{y} \geq f(\widehat{y})$). Consider the sequence $\{f^k(\overline{y})\}_{k=0}^\infty$ of elements of $[\overline{y}, \widehat{y}]$. By hypothesis of the theorem, there exists a subsequence $\{f^{k_n}(\overline{y})\}_{n=0}^\infty$ converging to an element of $[\overline{y}, \widehat{y}]$ which we denote \overline{x} . It is then easy to show that the existence of a convergent subsequence together with the property that the sequence is increasing imply that the whole sequence (and not only a subsequence) converges toward the unique \overline{x} . Since f is continuous, necessarily $f(\overline{x}) = \overline{x}$. Consequently, \overline{x} is a fixed point of f . Note that if x is an arbitrary fixed point of f in $[\overline{y}, \widehat{y}]$, $x \geq \overline{y}$ and for all $k = 1, \dots, n$ $x = f^k(x) \geq f^k(\overline{y})$ by monotonicity of f , so that $x \geq \overline{x}$ necessarily. This establishes that \overline{x} is the minimal fixed point of f in $[\overline{y}, \widehat{y}]$. Alternatively, consider the function $g : [-\widehat{y}, -\overline{y}] \rightarrow [-\widehat{y}, -\overline{y}]$ with $g(x) = -f(-x)$ and apply the same argument for \widehat{x} .

The hypothesized continuity of the mapping f in the theorem is sufficient to guarantee that the limit of the sequence $\{f^k(\overline{y})\}_{k=0}^\infty$ is a fixed point. This point is important, and the argument merits a detailed explanation. Continuity of f at \overline{x} means that, for all sequence $\{x^k\}_{k=0}^\infty$ converging to \overline{x} , the sequence $\{f(x^k)\}_{k=0}^\infty$ converges to $f(\overline{x})$. In particular, the sequence $\{f^k(\overline{y})\}_{k=0}^\infty$ converges to \overline{x} so that the sequence $\{f[f^k(\overline{y})]\}_{k=0}^\infty$ converges to $f(\overline{x})$. Obviously the two sequences coincide, and therefore $f(\overline{x}) = \overline{x}$. From this reasoning, it is obvious that continuity of f at \overline{x} is sufficient, but certainly not necessary. What is necessary for \overline{x} to be a fixed point is that $f(\overline{x}) = \overline{x}$, which we will prove in our setup without using the continuity argument.

3.1 Existence of Minimal and Maximal Fixed Points

We now construct a particular lattice E and an order interval of this lattice in which we apply our fix point theorem. Consider the space E of functions $h : \mathbb{X} = \mathbb{R}_+ \times \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$ satisfying the following conditions:

- (i). h is continuous;
- (ii). h is weakly increasing;
- (iii). $F - h$ is weakly increasing;

The space E is a lattice with the pointwise partial order \leq defined as $h \leq g$ if $g(x', y', z') \geq h(x, y, z)$ for all $(x', y', z') \geq (x, y, z)$ in \mathbb{X} , and the operations:

$$(h \vee g)(x, y, z) = \max\{h(x, y, z), g(x, y, z)\}$$

and

$$(h \wedge g)(x, y, z) = \min\{h(x, y, z), g(x, y, z)\}$$

are the lattice operations.⁹ In this space E , consider the set of functions c such that $0 \leq c \leq F$, that is, the order interval $[0, F]$. The following lemma establishes an important property of the interval $[0, F]$.¹⁰

Lemma 1. The order interval $[0, F]$ is an equicontinuous set of functions.

Proof. Equicontinuity is here induced by the double monotonicity of the elements of $[0, F]$ (properties (ii) and (iii) in conjunction with the uniform continuity of F). The assumption of uniform continuity of F on its domain implies that:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad |(x, y, z) - (x', y', z')| < \delta \implies |F(x', y', z') - F(x, y, z)| < \varepsilon$$

For all c in $[0, F]$, properties (ii) and (iii) imply that, for all $(x', y', z') \geq (x, y, z)$:

$$0 \leq c(x', y', z') - c(x, y, z) \leq F(x', y', z') - F(x, y, z)$$

Combining this last inequality with the uniform continuity of F leads to:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad |(x, y, z) - (x', y', z')| < \delta \implies \forall c \in [0, F], \quad |c(x', y', z') - c(x, y, z)| < \varepsilon$$

which demonstrates that $[0, F]$ is an equicontinuous set of functions.

Corollary. Every sequence in $[0, F]$ has a convergent subsequence that converges to an element in $[0, F]$.

The equicontinuous functions are defined on the separable metric space \mathbb{X} , so that the Arzela-Ascoli ([20], p.169) theorem applies. Consequently, any sequence $\{c_k\}_{k=0}^\infty$ in $[0, F]$ has a subsequence $\{c_{k_n}\}_{n=0}^\infty$ that converges pointwise to a continuous function and the convergence is uniform on all compact subsets of \mathbb{X} . Denote by \tilde{c} the limit of the subsequence. Obviously, $0 \leq \tilde{c} \leq F$, and, to verify that our fix point theorem apply, it remains to prove that \tilde{c} and $F - \tilde{c}$ are also weakly increasing. We use the fact that the convergence to \tilde{c} may not be uniform on the whole space \mathbb{X} but nevertheless preserves the monotonicity properties of the c_k and the $F - c_k$, and therefore equicontinuity.

More rigorously, consider any s in \mathbb{X} and pick any $s' \geq s$. By the Ascoli-Arzela theorem the subsequence $\{c_{k_n}\}_{n=0}^\infty$ converges uniformly in the compact space $[s, s']$. This implies that:

$$\forall \varepsilon > 0 \exists N > 0 \text{ for all } s \leq s'' \leq s' \text{ for all } n \geq N \quad |c_{k_n}(s'') - \tilde{c}(s'')| < \varepsilon \quad (4)$$

In particular,

$$|c_{k_n}(s) - \tilde{c}(s)| < \varepsilon$$

⁹Indeed the operations preserve continuity and also monotonicity. If both h and g are weakly increasing, then $\max(h, g)$ is also weakly increasing. Alternatively, if both $F - h$ and $F - g$ are increasing, then $F - \max(h, g) = \min(F - h, F - g)$ is also increasing.

¹⁰Note that $[0, F]$ is a sublattice of E .

Then, because all c_{k_n} are weakly increasing, either:

- (a). For all $n \geq N$, $c_{k_n}(s'') = c_{k_n}(s)$, and naturally $\tilde{c}(s'') = \tilde{c}(s)$ or,
- (b). There exists $n > N$ and $\phi > 0$ such that:

$$c_{k_n}(s'') \geq c_{k_n}(s) + 2\phi$$

Setting $\varepsilon = \phi$ in (6), we then have that $\tilde{c}(s'') \geq c_{k_n}(s'') - \phi \geq c_{k_n}(s) + \phi \geq \tilde{c}(s)$ so that $\tilde{c}(s'') \geq \tilde{c}(s)$ which proves that \tilde{c} is weakly increasing. A similar reasoning applies to $F - \tilde{c}$.

In the footsteps of Coleman [10] we define the mapping A on $[0, F]$ from the Euler equation (3) as follows:

$$\begin{aligned} u'((Ac)(K, z)) &= \beta E_z \{ u' [c(F(K, z) - (Ac)(K, z), z')] \\ &\quad \cdot H[F(K, z) - (Ac)(K, z), z'] \} \end{aligned}$$

so that any fixed point of A is an equilibrium consumption function. Note that 0 is a fixed point of A (i.e., $A0 = 0$), and that $AF \leq F$.

Proposition. Under Assumption 1-2, for any c in $[0, F]$, a unique $A(c)$ in $[0, F]$ exists. Furthermore, the operator A is monotone.

Proof. The proof that Ac exists, is unique, continuous, weakly increasing, and that $F - Ac$ is weakly increasing follows the construction in Coleman (1991), as does the monotonicity of A . Specifically, considering c_1 and c_2 such that $c_1 \leq c_2$, we have:

$$\begin{aligned} u'[Ac_2(K, z)] &= \beta E_z \{ u' [c_2(F(K, z) - Ac_2(K, z))] \\ &\quad \cdot H(F(K, z) - Ac_2(K, z)) \} \end{aligned}$$

and:

$$u'[Ac_2(K, z)] \leq \beta E_z \{ u' [c_1(F(K, z) - Ac_2(K, z))] H(F(K, z) - Ac_2(K, z)) \} \quad (5)$$

Assume that $Ac_1 \geq Ac_2$. Then, $Ac_1(K, z) \geq Ac_2(K, z)$ and $F(K, z) - Ac_2(K, z) \geq F(K, z) - Ac_1(K, z)$. Because c_1 is increasing:

$$c_1(F(K, z) - Ac_1(K, z)) \leq c_1(F(K, z) - Ac_2(K, z))$$

With both u' and H decreasing functions, the previous inequality implies that:

$$\begin{aligned} u'[Ac_1(K, z)] &= \beta E_z \{ u' [c_1(F(K, z) - Ac_1(K, z))] H(F(K, z) - Ac_1(K, z)) \} \\ &\geq \end{aligned} \quad (6)$$

$$\beta E_z \{ u' [c_1(F(K, z) - Ac_2(K, z))] H(F(K, z) - Ac_2(K, z)) \}$$

Combining (5) and (6) leads to:

$$u'[Ac_1(K, z)] \geq u'[Ac_2(K, z)]$$

which contradicts the hypothesis that $Ac_1 \geq Ac_2$. It must therefore be that $Ac_1 \leq Ac_2$, that is, A is a monotone operator.

Suppose there exists \bar{y} such that $A\bar{y} \geq \bar{y}$, and consider the sequence $c_0 = \bar{y}$ and $c_{n+1} = Ac_n$ for all n . By Corollary 1, the sequence $\{c_k\}_{k=0}^\infty$ has a subsequence that converges to a limit \tilde{c} in $[0, F]$. Furthermore, because the sequence $\{c_k\}_{k=0}^\infty$ is increasing, the whole sequence converges to the limit \tilde{c} . Finally, and as noted above, for our theorem to apply it is sufficient to show that $A\tilde{c} = \tilde{c}$.

Proposition. *The limit \tilde{c} of the sequence $\{A^k(\bar{y})\}_{k=0}^\infty$ is a fixed point of the operator A , that is, $A\tilde{c} = \tilde{c}$.*

Proof. Pick any $K = (x, y)$ in $\mathbb{R}_+ \times \mathbb{R}_+$ and consider $s = (K, z)$ for any $z \in Z$. We demonstrate that $A\tilde{c} = \tilde{c}$ pointwise. Assume, without loss of generality, that $x \geq y$. The sequence $\{c_{n+1}\}_{n=0}^\infty = \{Ac_n\}_{n=0}^\infty$ converges to \tilde{c} pointwise, so that:

$$F(s) - Ac_n(s) \text{ converges to } F(s) - \tilde{c}(s)$$

and, since H is continuous:

$$H(F(s) - Ac_n(s)) \text{ converges to } H(F(s) - \tilde{c}(s)) \quad (7)$$

We also know that the convergence of sequence $\{c_n\}_{n=0}^\infty$ toward \tilde{c} is uniform on the compact space $Y = [0, F(x, x, z^{\max})] \times [0, F(x, x, z^{\max})] \times Z$. Consequently:

$$c_n(F(s) - Ac_n(s)) \text{ converges to } \tilde{c}(F(s) - \tilde{c}(s))$$

Note that the uniform convergence toward \tilde{c} is essential in establishing this result. Indeed:

$$|c_n(F(s) - Ac_n(s)) - \tilde{c}(F(s) - \tilde{c}(s))|$$

$$\leq$$

$$|c_n(F(s) - Ac_n(s)) - \tilde{c}(F(s) - Ac_n(s))|$$

$$+$$

$$|\tilde{c}(F(s) - Ac_n(s)) - \tilde{c}(F(s) - \tilde{c}(s))|$$

The first absolute value on the right side of the inequality above is bounded above by $\sup |c_n - \tilde{c}|$ on the compact Y , which can be made arbitrarily small

because of the uniform convergence on the compact Y . The second absolute value can be made arbitrarily small by equicontinuity of \tilde{c} .

Then, by continuity of u' , for all z in Z :

$$u'[c_n(F(s) - Ac_n(s))] \text{ converges to } u'[\tilde{c}(F(s) - \tilde{c}(s))] \quad (8)$$

Thus, combining (7) and (8), which are satisfied for $k = (x, y)$ and any $z \in Z$::

$$\begin{aligned} & \beta E_z \{u'[c_n(F(s) - Ac_n(s))]H(F(s) - Ac_n(s))\} \\ & \text{converges to } \beta E_z \{u'[\tilde{c}(F(s) - \tilde{c}(s))]H(F(s) - \tilde{c}(s))\} \end{aligned}$$

The term on the left is exactly $u'(Ac_n(s))$ which we know converges to $u'(\tilde{c}(s))$. By uniqueness of the limit, it must be that:

$$u'(\tilde{c}(s)) = \beta u'[\tilde{c}(F(s) - \tilde{c}(s))]\tilde{c}(F(s) - \tilde{c}(s))$$

which demonstrates that, pointwise, $A\tilde{c} = \tilde{c}$.

A completely symmetric reasoning applies for the sequence defined by $c_0 = \hat{y}$ and $c_{n+1} = Ac_n$ in which \hat{y} is an excessive point, i.e. $A\hat{y} \leq \hat{y}$ (for instance, $\hat{y} = F$). We can now state the main proposition of this section of the paper.

Proposition. Under Assumption 1-2, there exists a maximal fixed point of A in $[0, F]$, which can be obtained as the limit of the sequence $A^n(F)$.

Proof: The proof is a direct application of the theorem demonstrated at the beginning of this Section.

3.2 A Refinement of the Maximal Fixed Point

Consider the sequence of policies $\{A^n F\}_{n \in \mathbb{N}}$, which, we have established in the previous proposition, converges to the maximal fixed point, denoted \tilde{c} . Consider then the sequence of value functions $\{\hat{v}_n\}_{n=0}^\infty$ generated from the following recursion:

$$\hat{v}_n(k, K, z) = \sup_{c \in \Gamma(k, K, z)} \{u(c) + \quad (9)$$

$$\beta E_z[\hat{v}_{n-1}(f(k, K, z) - c, F(K, z) - A^{n-1}F(K, z), z')]\} \quad (10)$$

and with $\hat{v}_0 \equiv 0$. Our strategy is to demonstrate that the sequence \hat{v}_n converges to the solution v of Bellman's equation associated with the household's maximization problem. If $\{\hat{v}_n\}_{n=0}^\infty$ converges to v , since by construction the optimal policy function maximizing the right side of the previous equality, evaluated along the equilibrium path, is exactly $A^{n-1}F(K, z)$, then by Theorem 9.9 in Stokey, Lucas, with Prescott [22], the sequence of functions $A^{n-1}F(K, z)$ converges pointwise to the optimal policy associated with v , which we have demonstrated must be strictly positive in Section 2. We now show that the above stated convergence is true.

Notice first that the sequence $\{\hat{v}_n\}_{n=0}^\infty$ is convergent. To demonstrate this property, define the operator T_n as follows:

$$(T_{n-1}\hat{v}_{n-1})(k, K, z) = \sup_{c \in \Gamma(k, K, z)} \{u(c) + \beta E_z[\hat{v}_{n-1}(f(k, K, z) - c, F(K, z) - A^{n-1}F(K, z), z')]\}$$

for $n \geq 1$, and $\hat{v}_n = T_{n-1}\hat{v}_{n-1}$. Obviously, each T_j is a contraction of modulus $\beta < 1$ so that the sequence $T_n \circ T_{n-1} \circ \dots \circ T_0(v_0)$ is a Cauchy sequence, and therefore converges to a unique limit.

Second, applying the same argument as in Greenwood and Huffman [15] establishes that the sequence $\{\hat{v}_n\}_{n=0}^\infty$ converges to v on any compact subset of the state space.

Together these two results imply that $\lim_{n \rightarrow \infty} \{\hat{v}_n\}_{n=0}^\infty = v$.

4 UNIQUENESS OF EQUILIBRIUM

This section establishes uniqueness of equilibrium under fairly standard assumptions by following a method similar to the one in Coleman [11] and Datta, Mirman, and Reffett [12]. Specifically, we demonstrate that any fixed point of the operator A is also a fixed point of another operator \hat{A} , and that \hat{A} has at most one interior fixed point because it is pseudo concave and x_0 -monotone, a result established by Coleman [10]. We remind the reader of this result, and we slightly amend the proof of Coleman [10] to address the case of a non-compact state space.

Theorem. *An operator \hat{A} that is pseudo concave and x_0 -monotone has at most one strictly positive fixed point.*

Proof. Suppose that \hat{A} has two distinct strictly positive fixed points, which we denote c_1 and c_2 . Assume without loss of generality that there exists (\hat{k}, \hat{z}) with $\hat{k} > 0$ such that $c_1(\hat{k}, \hat{z}) < c_2(\hat{k}, \hat{z})$. Choose $0 < k_1 \leq \hat{k}$ and $0 < t < 1$ such that:

$$c_1(k, z) \geq tc_2(k, z) \text{ for all } k_1 \leq k \leq \sup(\hat{k}, 2k_1), \text{ all } z \quad (11)$$

with equality for some (k, z) . Note that such t exists because the interval $[k_1, \sup(\hat{k}, 2k_1)]$ is compact.¹¹ Combining the x_0 -monotonicity of \hat{A} and (12) implies:

$$c_1(k, z) \geq (\hat{A}tc_2)(k, z) \text{ for all } k_1 \leq k \leq \sup(\hat{k}, 2k_1), \text{ all } z$$

We therefore have that, for all z and for all $k_1 \leq k \leq \sup(\hat{k}, 2k_1)$:

$$c_1(k, z) \geq \hat{A}tc_2(k, z) > t\hat{A}c_2(k, z) = tc_2(k, z)$$

¹¹In Coleman, the existence of t such that $c_1 \geq tc_2$ is guaranteed because the strictly positive consumption functions are compared on the compact set $[k_1, \bar{k}] \times Z$ where \bar{k} is the maximum maintainable capital-labor ratio.

in which the strict inequality, which follows from the hypothesis of pseudo concavity of \hat{A} , contradicts (12). Therefore, there is at most one fixed point.¹²

We construct the operator \hat{A} as follows. First define the set of functions $m : \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$ such that:

- (i). m is continuous,
- (ii). For all $(K, z) \in \mathbb{R}_+ \times Z$, $0 \leq m(K, z) \leq F(K, z)$
- (iii). For any $K = 0$, $m(K, z) = 0$.

Denote M this set, which is endowed with the standard pointwise partial ordering. Consider the function $\Psi(m(K, z))$ implicitly defined by:

$$u'[\Psi(m(K, z))] = \frac{1}{m(K, z)} \text{ for } m > 0, 0 \text{ elsewhere}$$

Naturally, Ψ is continuous, increasing, $\lim_{m \rightarrow 0} \Psi(m) = 0$, and $\lim_{m \rightarrow F(K, z)} \Psi(m) = F(K, z)$. Using the function Ψ , we denote:

$$\hat{Z}(m, \tilde{m}, K, z) = \frac{1}{\tilde{m}} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(\tilde{m}(K, z)), z')}{m(F(K, z) - \Psi(\tilde{m}(K, z)), z')} \right\}$$

and consider the operator \hat{A} :

$$\hat{A}m = \{\tilde{m} \mid \hat{Z}(m, \tilde{m}, K, z) = 0 \text{ for } m > 0, 0 \text{ elsewhere}\}$$

Since \hat{Z} is strictly increasing in m and strictly decreasing in \tilde{m} , and since $\lim_{\tilde{m} \rightarrow 0} \hat{Z} = +\infty$ and $\lim_{\tilde{m} \rightarrow F(K, z)} \hat{Z} = -\infty$, for each $m(K, z) > 0$, with $K > 0$, and $z \in Z$ there exists a unique $\hat{A}m(K, z)$.

Note that we can relate each orbit of the operator A to a specific orbit of the operator \hat{A} in the following manner. Given any c_0 in the order interval $[0, F]$ of E , there exists a unique m_0 in M such that:

$$m_0(K, z) = \frac{1}{u'(c_0(K, z))}$$

By construction, there exists a unique $\hat{A}m_0$ that satisfies $\hat{Z}(m_0, \hat{A}m_0, K, z) = 0$, that is:

$$\frac{1}{\hat{A}m_0(K, z)} = \beta E_z \left\{ \frac{H(F(K, z) - \Psi(\hat{A}m_0(K, z)), z')}{m_0(F(K, z) - \Psi(\hat{A}m_0(K, z)), z')} \right\}$$

or, equivalently (from the definition of c_0):

$$\begin{aligned} \frac{1}{\hat{A}m_0(K, z)} &= \beta E_z \{ H(F(K, z) - \Psi(\hat{A}m_0(K, z)), z') \\ &\quad \cdot u'(c_0(F(K, z) - \Psi(\hat{A}m_0(K, z)), z')) \} \end{aligned}$$

¹²Note that the hypothesis that \hat{A} is monotone is not necessary.

By construction, Ac_0 satisfies:

$$\begin{aligned} u'((Ac_0)(K, z)) &= \beta E_z \{ H(F(K, z) - Ac_0(K, z), z') \\ &\quad \cdot u'(c_0(F(K, z) - Ac_0(K, z), z')) \} \end{aligned}$$

Therefore, by uniqueness of $\hat{A}m_0$ it must be that $1/\hat{A}m_0 = u'(Ac_0)$ (or, equivalently, that $\Psi(\hat{A}m_0) = Ac_0$). By induction, it is trivial to demonstrate that for all $n = 1, 2, \dots$ $A^n c_0 = \Psi(\hat{A}^n m_0)$.

It is easy to show that to each fixed point of the operator A corresponds a fixed point of the operator \hat{A} . Indeed, consider x such that $Ax = x$ and define $y = 1/u'(x)$ (or, equivalently $\Psi(y) = x$). By definition, x satisfies:

$$\begin{aligned} u'(x(K, z)) &= \beta E_z \{ H(F(K, z) - x(K, z), z') \\ &\quad \cdot u'(x(F(K, z) - x(k, z), z')) \} \text{ for all } (K, z) \end{aligned}$$

Substituting the definition of y into this expression, this implies that:

$$\frac{1}{y} = \beta E_z \frac{H(F(K, z) - \Psi(y(K, z)), z')}{y(F(K, z) - \Psi(y(k, z), z'))}$$

which shows that y is a fixed point of \hat{A} . The following proposition, in conjunction with the theorem demonstrated in the beginning of this section, implies that \hat{A} has at most one fixed point. Thus, A also has at most one fixed point, although at least one ($\lim_{n \rightarrow \infty} A^n F$). Therefore, A has exactly one fixed point.

Proposition. The operator \hat{A} is pseudo concave and x_0 -monotone.

Proof: Recall that \hat{A} is pseudo concave if, for any strictly positive m and any $0 < t < 1$, $\hat{A}tm(K, z) > t\hat{A}m(K, z)$ for all $K > 0$ and for all $z \in Z$. Since \hat{Z} is strictly decreasing in its second argument, a sufficient condition for this to be true is that:

$$\hat{Z}(tm, t\hat{A}m, K, z) > \hat{Z}(tm, \hat{A}tm, K, z) = 0 \quad (12)$$

By definition:

$$\hat{Z}(tm, t\hat{A}m, K, z) = \frac{1}{t\hat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(t\hat{A}m(K, z)), z')}{tm(F(K, z) - \Psi(t\hat{A}m(K, z)), z')} \right\}$$

so that:

$$t\hat{Z}(tm, t\hat{A}m, K, z) = \frac{1}{\hat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(t\hat{A}m(K, z)), z')}{m(F(K, z) - \Psi(t\hat{A}m(K, z)), z')} \right\}$$

Since Ψ is increasing and $H(K', z')/m(K', z')$ is decreasing in K' :

$$\frac{1}{\hat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(t\hat{A}m(K, z)), z')}{m(F(K, z) - \Psi(t\hat{A}m(K, z)), z')} \right\}$$

>

$$\frac{1}{\widehat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(\widehat{A}m(K, z)), z')}{m(F(K, z) - \Psi(\widehat{A}m(K, z)), z')} \right\} = 0$$

and $\widehat{Z}(tm, t\widehat{A}m, K, z) > 0$ so that condition (10) obtains.

The condition that $\lim_{k \rightarrow 0} f_1(k, K, z) = \infty$ for all $K > 0$, all z in Assumption 1 (ii) implies that $H(0, z') = \infty$ for all z' . Given that \widehat{A} is monotone, this latter condition is sufficient for the operator \widehat{A} to be x_0 -monotone (Lemma 9 and 10 in Coleman [10]).

5 CONCLUDING REMARKS

This paper provides an extension to the work of Coleman [10][11] and Greenwood and Huffman [15] to the case of unbounded growth. Such an extension is important, as many models studied in the applied growth and macroeconomics literature are formulated on unbounded state spaces. We establish the competitive equilibrium as the unique fixed point of a mapping. This extension is not trivial, since all the standard fixed point results used in the literature do not apply to our problem, because relaxing the assumption of a compact state space makes it very difficult to establish suitable algebraic, analytic, or order structures on particular spaces of functions. Consequently, basic or general fixed point theorems cannot be applied. However, the mapping corresponding to the recursive problem -expressed in the form of iterations on an operator defined via an Euler equation- is well defined and has critical monotonic properties. Our strategy is then to directly demonstrate the existence of a fixed point of this mapping by producing this fixed point as the limit of a simple algorithm.

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