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Abstract

Ray (1998) developed measures of input- and output-oriented scale efficiency that can be directly computed from an estimated Translog frontier production function. This note extends the earlier results from Ray (1998) to the multiple-output multiple input case.

MEASURING SCALE EFFICIENCY FROM THE TRANSLOG MULTI-INPUT, MULTI-OUTPUT DISTANCE FUNCTION

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Ray (1998) developed measures of input- and output-oriented scale efficiency that can be directly computed from an estimated Translog frontier production function. The frontier production function is defined only for the single output case. Typically, multiple output technologies are represented by cost or profit functions. In recent years, however, the multiple-output multiple-input Distance function is gaining acceptance as an alternative analytical format in applied production analysis (e.g., Coelli and Perelman (1999) and Atkinson and Primont (2002)). This note extends the earlier results from Ray (1998) to the multiple-output multiple input case.

The Theoretical Background:

Consider an n -input, m -output production technology. An input-output bundle (x^0, y^0) is a feasible production plan if the output vector y^0 is producible from the input vector x^0 . The production possibility set can be defined as

$$T = \{(x, y) : y \text{ can be produced from } x\}. \quad (1)$$

Following Shephard (1953) one can define the input-oriented Distance function evaluated at (x, y) as

$$D^I(x, y) = \max \lambda : (\frac{1}{\lambda}x, y) \in T. \quad (2)$$

The following properties of the input-oriented Distance function ¹should be noted:

- (a) $D^I(x, y)$ is homogeneous of degree -1 in x ;

¹ For the properties of input- and output-oriented Distance functions, see Färe, Grosskopf, and Lovell (1994) and Färe and Primont (1995).

(b) $D^I(x, y)$ is decreasing in x .

Similarly, the output-oriented Distance function is

$$D^O(x, y) = \min \mu : (x, \frac{1}{\mu} y) \in T. \quad (3)$$

Analogous to the input-oriented Distance function the following properties apply to the output-oriented Distance function:

(c) $D^O(x, y)$ is homogeneous of degree 1 in y ;

(d) $D^O(x, y)$ is increasing in y .

Under the standard assumption of free disposability of inputs and outputs, two alternative and equivalent ways to characterize the production possibility set would be

$$T = \{(x, y) : D^I(x, y) \geq 1\} \quad (4a)$$

and

$$T = \{(x, y) : D^O(x, y) \geq 1\}. \quad (4b)$$

The graph of the technology is the set

$$G = \{(x, y) : D^I(x, y) = 1\} = \{(x, y) : D^O(x, y) = 1\}. \quad (5)$$

Consider an input-output bundle (x^0, y^0) and suppose $D^I(x^0, y^0) = \rho$. Thus,

$\frac{1}{\rho} D^I(x^0, y^0) = 1$. Define $\bar{x}^0 = \frac{1}{\rho} x^0$. Then, by property (a), $D^I(\bar{x}^0, y^0) = 1$. Thus, $\ln D^I(\bar{x}^0, y^0) = 0$.

Next, consider the set

$$S = \{(t_1, t_2) : D^I(t_1 \bar{x}^0, t_2 y^0) = 1\}. \quad (6)$$

Following Banker (1984), we can define the input-output pair $(x_*^0, y_*^0) = (t_1^* \bar{x}^0, t_2^* y^0)$ as a most productive scale size (MPSS) if

$$\frac{t_2^*}{t_1^*} \geq \frac{t_2}{t_1} \text{ for all } (t_1, t_2) \in S.$$

Banker (1984) has shown that locally constant returns to scale (CRS) hold at an MPSS.

Now, under CRS, both the input- and the output-oriented Distance function is

homogeneous of degree 0 in (x, y) . Thus, at (x_*^0, y_*^0)

$$\sum_i \frac{\partial D^I}{\partial x_i} x_i + \sum_j \frac{\partial D^I}{\partial y_j} y_j = 0. \quad (7a)$$

Further, because $D^I(x, y) = 1$ at this point,

$$\sum_i \frac{\partial \ln D^I}{\partial \ln x_i} + \sum_j \frac{\partial \ln D^I}{\partial \ln y_j} = 0 \text{ at } (x_*^0, y_*^0). \quad (7b)$$

Define

$$\frac{\partial \ln D^I}{\partial \ln x_i} = \varepsilon_i^x, \frac{\partial \ln D^I}{\partial \ln y_j} = \varepsilon_j^y, \sum_i \varepsilon_i^x = \varepsilon^x, \text{ and } \sum_j \varepsilon_j^y = \varepsilon^y.$$

Thus, at the MPSS, (x_*^0, y_*^0) ,

$$\varepsilon^x + \varepsilon^y = 0. \quad (8)$$

Further, by virtue of property (a), $\varepsilon^x = -1$ at every input bundle x . Hence, at the MPSS,

$$\varepsilon^y = 1.$$

The Input-oriented Translog Distance Function

Consider the input-oriented Translog Distance function

$$\begin{aligned} \ln D^I = & \alpha_0 + \sum_i \alpha_i \ln x_i + \frac{1}{2} \sum_i \sum_j \alpha_{ij} \ln x_i \ln x_j + \sum_k \beta_k \ln y_k + \sum_k \sum_l \beta_{kl} \ln y_k \ln y_l \\ & + \sum_i \sum_k \gamma_{ik} \ln x_i \ln y_k. \end{aligned} \quad (9)$$

From the above,

$$\varepsilon_i^x = \alpha_i + \sum_j \alpha_{ij} \ln x_j + \sum_k \gamma_{ik} \ln y_k. \quad (10)$$

Because $\sum_i \varepsilon_i^x = \varepsilon^x = -1$ for all pairs (x, y) , we get the parameter restrictions

$$\sum_i \alpha_i = -1; \sum_j \alpha_{ij} = 0, \sum_k \gamma_{ik} = 0 \text{ for all } i. \quad (11)$$

Finding the MPSS from the input-oriented Distance Function

Evaluated at (x_*^0, y_*^0) the Translog input-oriented Distance function is

$$\begin{aligned} \ln D^I(x_*^0, y_*^0) &= \alpha_0 + \sum_i \alpha_i (\ln \bar{x}_{0i} + \ln t_1^*) + \frac{1}{2} \sum_i \sum_j \alpha_{ij} (\ln \bar{x}_{0i} + \ln t_1^*) (\ln \bar{x}_{0j} + \ln t_1^*) + \\ &\sum_k \beta_k (\ln y_{0k} + \ln t_2^*) + \frac{1}{2} \sum_k \sum_l \beta_{kl} (\ln y_{0k} + \ln t_2^*) (\ln y_{0l} + \ln t_2^*) \\ &+ \sum_i \sum_k \gamma_{ik} (\ln \bar{x}_{0i} + \ln t_1^*) (\ln y_{0k} + \ln t_2^*). \end{aligned} \quad (12)$$

This can be simplified as

$$\begin{aligned} \ln D^I(x_*^0, y_*^0) &= \alpha_0 + \sum_i \alpha_i \ln \bar{x}_i + \frac{1}{2} \sum_i \sum_j \alpha_{ij} \ln \bar{x}_{0i} \ln \bar{x}_{0j} + \sum_k \beta_k \ln y_{0k} + \sum_k \sum_l \beta_{kl} \ln y_{0k} \ln y_{0l} \\ &+ \sum_i \sum_k \gamma_{ik} \ln \bar{x}_{0i} \ln y_{0k} + (\ln t_1^*) \left[\sum_i \left(\alpha_i + \sum_j \alpha_{ij} \ln x_j + \sum_k \gamma_{ik} \ln y_k \right) \right] + \\ &(\ln t_2^*) \left[\sum_k \left(\beta_k + \sum_l \beta_{kl} \ln y_{0l} + \sum_l \gamma_{il} \ln y_{0l} \right) \right] + \frac{1}{2} \left(\sum_k \sum_l \beta_{kl} \right) (\ln t_2^*)^2 + \left(\sum_i \sum_k \gamma_{ik} \right) \ln t_1^* \ln t_2^* \\ &= \ln D^I(\bar{x}^0, y^0) + \varepsilon^x(\bar{x}^0, y^0) \ln t_1^* + \varepsilon^y(\bar{x}^0, y^0) \ln t_2^* + \frac{1}{2} \left(\sum_k \sum_l \beta_{kl} \right) (\ln t_2^*)^2 + \left(\sum_i \sum_k \gamma_{ik} \right) \ln t_1^* \ln t_2^*. \end{aligned} \quad (13)$$

Recall, now, that $\ln D^I(\bar{x}^0, y^0) = 0$ and $\varepsilon^x(\bar{x}^0, y^0) = -1$. Also, $\sum_k \gamma_{ik} = 0$ for each i .

Define

$$\beta \equiv \sum_k \sum_l \beta_{kl}. \quad (14)$$

Then, because $\ln D^I(x_*^0, y_*^0) = 0$,

$$\ln t_1^* = \varepsilon^y(\bar{x}^0, y^0) \ln t_2^* + \frac{\beta}{2} (\ln t_2^*)^2. \quad (15)$$

Next by virtue of local CRS, at (x_*^0, y_*^0) ,

$$\begin{aligned} \varepsilon^y(x_*^0, y_*^0) &= \sum_k \left(\beta_k + \sum_l \beta_{kl} (\ln y_{0l} + \ln t_2^*) + \sum_i \gamma_{ik} (\ln \bar{x}_{0i} + \ln t_1^*) \right) \\ &= \varepsilon^y(\bar{x}^0, y^0) + \ln t_1^* \left(\sum_k \sum_i \gamma_{ik} \right) + \ln t_2^* \left(\sum_k \sum_l \beta_{kl} \right) \\ &= 1. \end{aligned} \quad (16)$$

Hence,

$$\ln t_2^* = \frac{(1 - \varepsilon^y(\bar{x}^0, y^0))}{\beta}. \quad (17)$$

Thus,

$$\begin{aligned} \ln t_1^* &= \left(\varepsilon^y(\bar{x}^0, y^0) + \frac{\beta}{2} (\ln t_2^*) \right) \ln t_2^* \\ &= \left(\frac{(1 + \varepsilon^y(\bar{x}^0, y^0))}{2} \right) \ln t_2^* = \frac{(1 - (\varepsilon^y(\bar{x}^0, y^0))^2)}{2\beta}. \end{aligned} \quad (18)$$

Therefore,

$$\ln t_1^* - \ln t_2^* = \frac{-(1 - \varepsilon^y(\bar{x}^0, y^0))^2}{2\beta} \quad (19)$$

and

$$\frac{t_1^*}{t_2^*} = e^{\frac{-(1 - \varepsilon^y(\bar{x}^0, y^0))^2}{2\beta}}. \quad (20)$$

Input-oriented Scale Efficiency

Consider the observed input-output bundle (x^0, y^0) . Suppose that x^0 represents one unit of a composite input while y^0 is one unit of a composite output. Then, by definition, ray average productivity at the observed input-output bundle is unity. The input-oriented efficient projection of (x^0, y^0) onto the graph of the technology is $(\bar{x}^0, y^0) = (\frac{1}{\rho} \bar{x}^0, y^0)$ where the ray average productivity is ρ . Finally, the input-oriented MPSS is

$(x_*^0, y_*^0) = (t_1^* \bar{x}^0, t_2^* y^0)$. Here t_2^* units of the composite output are produced from $\frac{t_1^*}{\rho}$ units of the composite input. Thus, the ray average productivity at the MPSS is

$$RAP^* = \frac{\rho t_2^*}{t_1^*}. \quad (21)$$

This also where the ray average productivity reaches a maximum.

The scale efficiency at the input-oriented efficient projection is

$$SE(\bar{x}^0, y^0) = \frac{RAP(\bar{x}^0, y^0)}{RAP^*} = \frac{t_1^*}{t_2^*} = e^{\frac{-(1-\varepsilon^y(\bar{x}^0, y^0))^2}{2\beta}}. \quad (22)$$

If $(\bar{x}^0, y^0)$ is itself an MPSS, $\varepsilon^y(\bar{x}^0, y^0)$ equal unity, scale efficiency is 1. Otherwise, this measure of scale efficiency lies between 0 and 1 so long as $\beta \equiv \sum_k \sum_l \beta_{kl} > 0$.

The Output-oriented Translog Distance Function

The output-oriented Translog Distance function can be specified as

$$\begin{aligned} \ln D^O(x, y) = & a_0 + \sum_i a_i \ln x_i + \frac{1}{2} \sum_i \sum_j a_{ij} \ln x_i \ln x_j + \sum_k b_k \ln y_k + \frac{1}{2} \sum_k \sum_l b_{kl} \ln y_k \ln y_l \\ & + \sum_i \sum_k g_{ik} \ln x_i \ln y_k. \end{aligned} \quad (23)$$

As in the case of the input-oriented Distance function, we can define

$$e_i^x = \frac{\partial \ln D^O}{\partial \ln x_i} = a_i + \sum_j a_{ij} \ln x_j + \sum_k g_{ik} \ln y_k, e^x = \sum_i e_i^x;$$

$$e_k^y = \frac{\partial \ln D^O}{\partial \ln y_k} = b_k + \sum_l b_{kl} \ln y_l + \sum_i g_{ik} \ln x_i, \text{ and } e^y = \sum_k e_k^y.$$

By virtue of the homogeneity property (d), $e^y = 1$ at every output bundle y . This leads to the parameter restrictions

$$\sum_k b_k = 1, \sum_l b_{kl} = 0 \text{ and } \sum_i g_{ik} = 0 \text{ for all } k. \quad (24)$$

Now suppose that $D^O(x^0, y^0) = \delta$. Define $\bar{y}^0 = \frac{1}{\delta} y^0$. Then $D^O(x^0, \bar{y}^0) = 1$.

Define the set

$$K = \{(k_1, k_2) : D^O(k_1 x^0, k_2 \bar{y}^0) = 1\}. \quad (25)$$

Suppose that $\frac{k_2^*}{k_1^*} \geq \frac{k_2}{k_1}$ for all $(k_1, k_2) \in K$. Then, $(\hat{x}^0, \hat{y}^0) = (k_1^* x^0, k_2^* \bar{y}^0)$ is an output-

oriented MPSS and is characterized by CRS locally. Hence,

$$\ln D^O(\hat{x}^0, \hat{y}^0) = 0 \text{ and } e^x(\hat{x}^0, \hat{y}^0) = -1.$$

In a manner analogous to the input-oriented case one can show that

$$\ln D^O(\hat{x}^0, \hat{y}^0) = \ln D^O(x^0, \bar{y}^0) + e^x(x^0, \bar{y}^0) \ln k_1^* + e^y(x^0, \bar{y}^0) \ln k_2^* + \frac{1}{2} \left(\sum_i \sum_j a_{ij} \right) (\ln k_1^*)^2$$

$$+ \left(\sum_i \sum_k g_{ik} \right) \ln k_1^* \ln k_2^* = 0. \quad (26)$$

Again, define $\sum_i \sum_j a_{ij} = \alpha$. Then,

$$\ln k_2^* = -(e^x(x^0, \bar{y}^0) + \frac{1}{2} \alpha \ln k_1^*) \ln k_1^*. \quad (27)$$

Next recall that at the MPSS $(\hat{x}^0, \hat{y}^0)$, $e^x = -1$. This implies that

$$e^x(\hat{x}^0, \hat{y}^0) = \sum_i \left(a_i + \sum_j a_{ij} (\ln x_{0j} + \ln k_1^*) + \sum_k g_{ik} (\ln \bar{y}_{0k} + \ln k_2^*) \right)$$

$$= e^x(x^0, \bar{y}^0) + \ln k_1^* \left(\sum_i \sum_j a_{ij} \right) + \ln k_2^* \left(\sum_i \sum_k g_{ik} \right) \quad (28)$$

$$= -1.$$

Again, by the homogeneity restriction, $\sum_i \sum_k g_{ik} = 0$.

Hence,

$$\ln k_1^* = -\frac{(1 + e^x(x^0, \bar{y}^0))}{\alpha}. \quad (29)$$

Substitution of (29) into (27) leads to

$$\ln k_2^* = \frac{(1 - e^x(x^0, \bar{y}^0))}{2} \ln k_1^*. \quad (30)$$

Thus,

$$\ln k_2^* - \ln k_1^* = \frac{(1 + e^x(x^0, \bar{y}^0))^2}{2\alpha}. \quad (31)$$

Hence,

$$\frac{k_1^*}{k_2^*} = e^{-\frac{(1 + e^x(x^0, \bar{y}^0))^2}{2\alpha}}. \quad (32)$$

Output-oriented Scale Efficiency

In terms of the composite input and output defined before, the ray average productivity at the observed input-output bundle is unity. At the efficient output-oriented projection onto the graph of the technology ray average productivity is

$$RAP(x^0, \bar{y}^0) = \frac{1}{\phi} \geq 1. \quad (33)$$

At the output-oriented MPSS, $(\hat{x}^0, \hat{y}^0)$, the ray average productivity is

$$RAP^* = \frac{k_2^*}{\phi k_1^*}. \quad (34)$$

Thus, scale efficiency at the efficient output-oriented projection is

$$SE(x^0, \bar{y}^0) = \frac{k_1^*}{k_2^*} = e^{\frac{-(1+e^x(x^0, \bar{y}^0))^2}{2\alpha}}. \quad (35)$$

As before, if the efficient projection is itself an MPSS, $e^x(x^0, \bar{y}^0) = -1$ and scale efficiency equals unity. Otherwise, $SE(x^0, \bar{y}^0) < 1$ so long as $\sum_i \sum_j a_{ij} = \alpha. > 0$.

Conclusion

This paper provides input- and output-oriented measures of scale efficiency that can be easily computed from an empirically estimated (appropriately oriented) multiple-input, multiple-output Translog Distance function. It may be emphasized that due to the linear homogeneity restrictions on the Distance functions, the scale elasticity measures (e^x or e^y) can be evaluated at the observed input-output bundle without having to adjust for technical inefficiency².

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² In am indebted to Catherine Morrison Paul for this insight.

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