



University of Connecticut

Department of Economics Working Paper Series

Continuous Paretian Preferences

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Working Paper 2003-29

August 2003

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Abstract

Two forms of continuity are defined for Pareto representations of preferences. They are designated continuity and coordinate continuity. Characterizations are given of those Pareto representable preferences that are continuously representable and, in dimension two, of those that are coordinate-continuously representable.

Journal of Economic Literature Classification: D11, D70

Keywords: preferences, Pareto order, continuous representations

1. Introduction

Debreu (1964) showed that every preference profile that is represented by a utility function is represented by a continuous utility function. The related question of continuous representability for a preference profile with a Pareto representation is addressed below.

Under the Pareto relation on $\mathfrak{R}^n$, one point is preferred to another if the points are distinct and the first equals or exceeds the second in every coordinate. Economists often define efficiency in terms of the Pareto relation; an efficient allocation of resources to n consumers is a feasible allocation to which no feasible allocation is Pareto preferred.

A Pareto representation for preferences over a set X is a function from X to $\mathfrak{R}^n$ that preserves both preference and indifference when $\mathfrak{R}^n$ is ordered by the Pareto relation. For example, consider a committee of n members whose preference profiles over a set X are represented by n utility functions. Suppose committee preferences over X are formed via unanimous pairwise voting. In other words one alternative is preferred to another if, when the committee votes on the pair, every member votes for the first alternative or abstains. Then the committee's preferences have a Pareto representation; the required function from X to $\mathfrak{R}^n$ has the n committee members' utility functions as its n coordinate functions.

More generally, a preference representation maps a given preference profile into a well-known, well-understood binary relation, such as greater than on $\mathfrak{R}$, the Pareto relation on $\mathfrak{R}^n$ or the lexicographic order on $\mathfrak{R}^n$.

There are at least three ways in which a *continuous* preference representation enhances our understanding of a preference profile above and beyond the level of understanding provided by a noncontinuous preference representation. First, a continuous representation preserves not only the order of alternatives, but also the proximity or closeness of alternatives, providing a clearer portrait of the given preference profile. Second a preference representation may enable a firm to turn consumer preferences into a demand function, or enable a government to calculate the social welfare effects of its actions, given individual preferences. Of course it is easier to calculate with continuous functions than with noncontinuous functions. Third, continuity can be helpful in determining whether a set of alternatives contains maximally preferred elements, and therefore, in the case of Pareto

representable preferences, whether a set of allocations contains any efficient allocations.

A Pareto representation will be called continuous if it is continuous as a function from X with the given order topology to $\Re^n$ with the Pareto order topology. It will be called coordinate continuous if its coordinate functions are continuous as functions from X with the given order topology to $\Re$ with the Euclidean topology. Theorems 1 and 2 present easy-to-apply tests for continuous representability and coordinate continuous representability of Paretian preferences: Pareto representable preferences are continuously representable if and only if every set of all alternatives indistinguishable from a given alternative is open; and Pareto representable preferences are coordinate-continuously representable if and only if every set of all alternatives indifferent to but distinguishable from a given alternative is open. (Two alternatives are distinguishable if there is an alternative that is preferred to exactly one of them, or to which exactly one of them is preferred).

Aizerman and Aleskerov (1995), Donaldson and Weymark (1998), Duggan (1999) and Knoblauch (2001) all discuss the existence of Pareto representations of preferences. More in the spirit of this paper, Sprumont (2001) characterizes preferences that are continuously Pareto representable. Sprumont's result differs from the results stated and proven below in three important ways: 1) Sprumont does not assume that the given preference relation is Pareto representable; 2) Sprumont assumes that the given set of alternatives is a compact connected subset of $\Re^n$, and he defines continuity in terms of the Euclidean topology on both the domain and range of the Pareto representation and 3) above and beyond the three key intermediateness conditions Sprumont's characterization includes four continuity conditions and three "richness" conditions, while Theorems 1 and 2 below require only one condition each. In short what makes Sprumont's result arguably the strongest result in the area is his formulation of the three intermediateness conditions that form the heart of his characterization of preferences Pareto representable in dimension two. My goal in stating and proving Theorems 1 and 2 below is to contribute a simple and clear handling of continuity by separating out the question of continuity from the question of Pareto representability.

Knoblauch (2002) discusses continuity and coordinate continuity for lexicographic preferences.

The paper is organized as follows. Section 2 contains definitions, a discussion of continuity, and the statements of two theorems characterizing continuously and coordinate continuously Pareto representable preferences. In Section 3 four examples illustrate the theorems. The theorems are proven in Section 4.

2. Two Existence Theorems.

A *binary relation* on a set X is a subset of $X \times X$. If $\succ$ is a binary relation on X , $\langle x, y \rangle \in \succ$ will be written $x \succ y$. For a binary relation $\succ$ on X , the associated binary relations *weak preference* and *indifference*, $\succeq$ and $\sim$, are defined by $x \succeq y$ if $\text{not}(y \succ x)$ and $x \sim y$ if $x \succeq y \succeq x$. If $x \in X$ let $W(x) = \{y \in X: x \succ y\}$ and let $B(x) = \{y \in X: y \succ x\}$.

The $\succ$ -*order topology* on X is the topology with subbasis consisting of all sets $W(x)$ and all sets $B(x)$.

A *Pareto relation* $>$ on $\mathbb{R}^n$ is defined by $x > y$ if $x \neq y$ and $x_i \geq y_i$ for all $i \in \{1, 2, \dots, n\}$. Then $x \geq y$ if and only if $\text{not}(y > x)$ if and only if $x = y$ or $x_i > y_i$ for some $i \in \{1, 2, \dots, n\}$.

A *Pareto representation* for a binary relation $\succ$ on a set X is a function $v: X \rightarrow \mathbb{R}^n$ such that, for $x, y \in X$, $x \succ y$ if and only if $v(x) > v(y)$.

A Pareto representation is *continuous* if it is continuous as a function from X with the $\succ$ -order topology to $\mathbb{R}^n$ with the Pareto order topology.

A topology on a set X is *discrete* if every subset of X is open.

A binary relation $\succ$ on X is *order discrete* if $\{y \in X: W(y) = W(x) \text{ and } B(y) = B(x)\}$ is open for every $x \in X$.

Theorem 1. *A Pareto representable binary relation has a continuous Pareto representation if and only if it is order discrete.*

More particularly, it will be seen in the proof that if $\succ$ has a Pareto representation $v: X \rightarrow \mathbb{R}^n$ and $n \geq 2$, then $\succ$ has a continuous Pareto representation $V: X \rightarrow \mathbb{R}^n$ if and only if $\succ$ is order discrete. Also, if $\succ$ has a Pareto representation $v: X \rightarrow \mathbb{R}^1$, then $\succ$ has a continuous representation $V: X \rightarrow \mathbb{R}^2$ if and only if $\succ$ is order discrete. However,

by Debreu [1964] order discreteness is not a necessary condition for the existence of a continuous Pareto representation $V: X \rightarrow \mathbb{R}^1$. See Example 4 below.

A binary relation $\succ$ on X is *weakly order discrete* if $\{y \in X: y \sim x \text{ and either } W(y) \neq W(x) \text{ or } B(y) \neq B(x)\}$ is open for every $x \in X$.

A preference representation $v: X \rightarrow \mathbb{R}^n$ for $\succ$ on X is *coordinate continuous* if for each $i \in \{1, 2, \dots, n\}$ v_i is a continuous function from X with the $\succ$ -order topology to $\mathbb{R}$ with the Euclidean topology.

Theorem 2. *A binary relation with a one- or two-dimensional Pareto representation has a coordinate continuous Pareto representation if and only if it is weakly order discrete.*

The following two propositions will serve as the basis for a comparison of the two forms of continuity.

Proposition 1. *A preference representation $v: X \rightarrow \mathbb{R}^n$ for $\succ$ on X is coordinate continuous if and only if it is continuous as a function from X with the $\succ$ -order topology to $\mathbb{R}^n$ with the Euclidean topology.*

Proof. The simple proof is given in Knoblauch [2002].

Proposition 2. *A continuous Pareto representation is coordinate continuous.*

Proof. Suppose $v: X \rightarrow \mathbb{R}^n$ is a continuous Pareto representation for $\succ$ on X , and $O \subseteq \mathbb{R}^n$ is open in the Euclidean topology. Since the Pareto-order topology is discrete, O is open in the Pareto-order topology. Since v is continuous, $v^{-1}(O)$ is open. This proves that v is continuous as a function from X to $\mathbb{R}^n$ with the Euclidean topology. By Proposition 1, v is coordinate continuous. ■

In light of Proposition 1 continuity seems to be a more appropriate concept than coordinate continuity for Pareto representations, since by Proposition 1 coordinate continuity

compares apples to oranges; coordinate continuity links the given order topology on X with the *strict* Pareto-order topology (the Euclidean topology) on $\mathbb{R}^n$, when the Pareto-order topology on $\mathbb{R}^n$ would be more appropriate for a Pareto representation.

An argument that coordinate continuity *is* appropriate can be made using the committee scenario of Section 1. Recall from Section 1 that a Pareto representation can be thought of as a committee of members whose preferences are generated by utility functions, with committee preferences formed by unanimous pairwise voting. Then coordinate continuity simply means members' utility functions are continuous. Here it is required that each utility function is continuous as a function from X with the given $\succ$ -order topology, not just as a function from X with the order topology generated by the member's own preferences.

Since this third form of continuity is uninteresting—there is little relationship between a given order topology and the order topology generated by a coordinate of a Pareto representation—and since it is also trivial in the sense that every coordinate of a Pareto representation can by Debreu (1964) always be taken to be continuous as a function from X with the order topology it generates, we will not further consider this third form of continuity.

Finally notice that when $n = 1$ all three forms of continuity are equivalent.

3. Four Examples.

The binary relation in Example 1 possesses neither a continuous nor a coordinate-continuous Pareto representation.

Example 1. Suppose $X = \{x, y, z\}$ and $z \sim x \succ y \sim z$. If $v: X \rightarrow \mathbb{R}^2$ is given by $v(x) = \langle 8, 3 \rangle$, $v(y) = \langle 7, 2 \rangle$ and $v(z) = \langle 5, 4 \rangle$, then v is a Pareto representation for $\succ$. However, $\{z\} = \{w \in X: w \sim x \text{ and either } W(w) \neq W(x) \text{ or } B(w) \neq B(x)\}$, but $\{z\}$ is not open, since the open sets of the $\succ$ -order topology on X are $\emptyset$, $\{x\}$, $\{y\}$, $\{x, y\}$ and $\{x, y, z\}$. Therefore $\succ$ is not weakly order discrete and, by Theorem 2, $\succ$ has no coordinate continuous representation. By Proposition 2, $\succ$ has no continuous Pareto representation.

The binary relation of Example 2 has a coordinate continuous, but not a continuous Pareto representation.

Example 2. Let $X = (0, 1) \times \{1\} \cup (2, 3) \times \{0\} \cup [4, 5] \times \{0\} \subseteq \mathbb{R}^2$ and define $\succ$ on X by $x \succ y$ if $x > y$. Clearly, $v(x) = x$ is a Pareto representation for $\succ$ that is neither continuous nor coordinate continuous. Notice that $\{< 1/2, 1 >\} = \{y \in X: W(y) = W(< 1/2, 1 >)\}$ and $B(y) = B(< 1/2, 1 >)\}$. Since $\{< 1/2, 1 >\}$ is not open in the $\succ$ -order topology on X , $\succ$ is not order discrete. By Theorem 1, $\succ$ has no continuous Pareto representation to $\mathbb{R}^n$ with $n \geq 2$.

If $x \in (0, 1) \times \{1\}$, then $\{y \in X: y \sim x \text{ and either } W(y) \neq W(x) \text{ or } B(y) \neq B(x)\} = (2, 3) \times \{0\} \cup [4, 5] \times \{0\}$, which is open in the $\succ$ -order topology on X . If $x \in (2, 3) \times \{0\} \cup [4, 5] \times \{0\}$ then $\{y \in X: y \sim x \text{ and either } W(y) \neq W(x) \text{ or } B(y) \neq B(x)\} = (0, 1) \times \{1\}$, which is open in the $\succ$ -order topology on X . Therefore $\succ$ is weakly order discrete and, by Theorem 2, $\succ$ has a coordinate Pareto representation. For example,

$$V(x) = \begin{cases} x & \text{if } x_1 < 3; \\ x - < 1, 0 > & \text{if } x_1 \geq 4. \end{cases}$$

is a coordinate-continuous Pareto representation for $\succ$.

The binary relation of Example 3 has a continuous, coordinate continuous Pareto representation.

Example 3. Let $X = \{< t, 4 - t >: 0 \leq t \leq 1 \text{ or } 3 \leq t \leq 4\} \cup \{< 1, 4 >, < 4, 1 >\} \subseteq \mathbb{R}^2$ and define $\succ$ on X by $x \succ y$ if $x > y$. Then $v(x) = x$ is a Pareto representation for $\succ$ and v is a coordinate continuous, but not continuous, since $\{< 4, 0 >\}$ is open in the Pareto order topology, but $v^{-1}(\{< 4, 0 >\}) = \{< 4, 0 >\}$ is not open in the $\succ$ -order topology.

Since $\{< 4, 1 >\} = \{y \in X: W(y) = W(< 4, 1 >) \text{ and } B(y) = B(< 4, 1 >)\}$, $\{< 1, 4 >\} = \{y \in X: W(y) = W(< 1, 4 >) \text{ and } B(y) = B(< 1, 4 >)\}$, $\{< t, 4 - t >: 0 \leq t \leq 1\} = \{y \in X: W(y) = W(< 0, 4 >) \text{ and } B(y) = B(< 0, 4 >)\}$, $\{< t, 4 - t >: 3 \leq t \leq 4\} = \{y \in X: W(y) = W(< 4, 0 >) \text{ and } B(y) = B(< 4, 0 >)\}$ and these four sets are open in the $\succ$ -order topology on X , $\succ$ is order discrete. By Theorem 1 there must exist a continuous Pareto representation for $\succ$.

Following the procedure that will be introduced in the proof of Theorem 1,

$$V(x) = \begin{cases} x & \text{if } x = \langle 4, 1 \rangle \text{ or } x = \langle 1, 4 \rangle; \\ \langle 0, 4 \rangle & \text{if } x \in \{\langle t, 4 - t \rangle : 0 \leq t \leq 1\}; \\ \langle 4, 0 \rangle & \text{if } x \in \{\langle t, 4 - t \rangle : 3 \leq t \leq 4\} \end{cases}$$

is a continuous Pareto representation for $\succ$. By Proposition 2, V is also coordinate continuous.

Example 4 illustrates the fact that Theorem 1 does not hold for $n = 1$. In particular, the binary relation of Example 4 possesses a continuous utility representation, but no continuous Pareto representation to $\Re^n$ with $n \geq 2$.

Example 4. Suppose $X = \Re$ and $\succ$ is defined by $x \succ y$ if $x > y$. Then $\{0\} = \{y \in X : W(y) = W(0) \text{ and } B(y) = B(0)\}$, but $\{0\}$ is not open in the $\succ$ -order topology. Therefore $\succ$ is not order discrete. By Theorem 1 $\succ$ has no continuous representation to $\Re^n$, $n \geq 2$. However, $v(x) = x$ is a continuous utility representation for $\succ$.

4. Proofs of Theorems.

Proof Theorem 1 ($\Rightarrow$). Suppose $\succ$ is a binary relation on X , $v : X \rightarrow \Re^n$ is a continuous Pareto representation for $\succ$, $n \geq 2$, $x \in X$ and $S = \{y \in X : W(y) = W(x) \text{ and } B(y) = B(x)\}$.

The set $v(S)$ is open, since the Pareto order topology on $\Re^n$ is discrete for $n \geq 2$. Since v is continuous, $v^{-1}(v(S))$ is open

Next $S = v^{-1}(v(S))$: if $y \in S$, then $y \in v^{-1}(v(S))$. If $y \notin S$ and $w \in S$, then $W(y) \neq W(w)$ or $B(y) \neq B(w)$. Without loss of generality, assume $W(y) \neq W(w)$. Then there exists $z \in X$ such that $w \succ z \succsim y$ or $y \succ z \succsim w$. Therefore $v(w) > v(z) \geq v(y)$ or $v(y) > v(z) \geq v(w)$. In either case, $v(y) \neq v(w)$. Since $v(y) \neq v(w)$ for all $w \in S$, $y \notin v^{-1}(v(S))$.

Since $S = v^{-1}(v(S))$ and $v^{-1}(v(S))$ is open, S is open. Therefore $\succ$ is order discrete.

■

Proof of Theorem 1 ($\Leftarrow$).

Suppose $\succ$ is a binary relation on X , $v: X \rightarrow \mathbb{R}^n$ is a Pareto representation for $\succ$, and $\succ$ is order discrete.

The binary relation $\approx$ on X defined by $x \approx y$ if $W(x) = W(y)$ and $B(x) = B(y)$ is an equivalence relation and therefore partitions X into a collection $\{X^\alpha\}$ of subsets of X . For each X^α choose a representative $x^\alpha \in X^\alpha$. Define $V: X \rightarrow \mathbb{R}^n$ by setting $V(x) = v(x^\alpha)$ for that α such that $x \in X^\alpha$.

For $x \in X^\alpha$ and $y \in X^\beta$,
 $x \succ y$ if and only if $x^\alpha \succ y$
 if and only if $x^\alpha \succ y^\beta$
 if and only if $v(x^\alpha) > v(y^\beta)$
 if and only if $V(x) > V(y)$

Therefore V is a Pareto representation for $\succ$.

Suppose $O \subseteq \mathbb{R}^n$. If $x \in V^{-1}(O)$, $y \in X$, $W(x) = W(y)$ and $B(x) = B(y)$, then $x \approx y$ so that $V(x) = V(y)$ by the definition of V . Therefore $y \in V^{-1}(O)$. In other words $V^{-1}(O) = \cup_{x \in V^{-1}(O)} \{y \in X: W(y) = W(x) \text{ and } B(y) = B(x)\}$. Since $\succ$ is order discrete, each set in this union is open, so that $V^{-1}(O)$ is open.

Since the inverse image under V of every subset of $\mathbb{R}^n$ is open, V is continuous. ■

Proof of Theorem 2 (n=1). Suppose $v: X \rightarrow \mathbb{R}^1$ is a Pareto representation for $\succ$ on X . Then v is a utility function for $\succ$, so that by Debreu [1964] $\succ$ has a continuous utility representation, that is, a coordinate continuous Pareto representation. Also, since v is a utility function representing $\succ$, $y \sim x$ implies $v(y) = v(x)$, so that $\{y \in X: y \sim x \text{ and either } W(y) \neq W(x) \text{ or } B(y) \neq B(x)\} = \emptyset$ for every $x \in X$. Therefore $\succ$ is weakly order discrete.

This establishes that, trivially, a Pareto representable binary relation is coordinate continuously representable if and only if it is weakly order discrete.

Proof of Theorem 2 (n=2) ($\Rightarrow$). Suppose $\succ$ is a binary relation on X and $v: X \rightarrow \mathbb{R}^2$ is a coordinate continuous Pareto representation for $\succ$. By Proposition 1, v is a continuous function from X with the $\succ$ -order topology to $\mathbb{R}^2$ with the Euclidean topology.

If $x, y \in X$ and $v(x) \neq v(y)$ then there exists $O \subseteq \mathbb{R}^2$ open in the Euclidean topology with $v(x) \in O$, $v(y) \in \mathbb{R}^2 - O$. Then $v^{-1}(O)$ is open, $x \in v^{-1}(O)$ and $y \in X - v^{-1}(O)$. Therefore there exists $z \in X$ such that $y \succsim z \succ x$ or $x \succ z \succsim y$. It follows that $W(x) \neq W(y)$ or $B(x) \neq B(y)$.

If $x, y \in X$ and $v(x) = v(y)$, then for $z \in X$, $z \in B(x)$ if and only if $v(z) > v(x)$ if and only if $v(z) > v(y)$ if and only if $z \in B(y)$. Therefore, $B(x) = B(y)$ and similarly $W(x) = W(y)$.

Combining the two arguments above, for $x, y \in X$

$$v(x) = v(y) \text{ if and only if } W(y) = W(x) \text{ and } B(y) = B(x) \quad (1)$$

By (1) and the fact that $y \sim x$ if and only if $v(y) \geq v(x) \geq v(y)$ (recall that $v(y) \geq v(x)$ means $v(x)$ is not Pareto preferred to $v(y)$), for every $x \in X$, $\{y \in X: y \sim x \text{ and either } W(y) \neq W(x) \text{ or } B(y) \neq B(x)\} = v^{-1}(\{s \in \mathbb{R}^2: s \geq v(x) \geq s \text{ and } s \neq v(x)\})$. Since $\{s \in \mathbb{R}^2: s \geq v(x) \geq s \text{ and } s \neq v(x)\}$ is open in the Euclidean topology and v is continuous from X with the $\succ$ -order topology to $\mathbb{R}^2$ with the Euclidean topology, $\{y \in X: y \sim x \text{ and either } W(y) \neq W(x) \text{ or } B(y) \neq B(x)\}$ is open. This proves that $\succ$ is weakly order discrete.

■

Proof of Theorem 2 (n=2) ($\Leftarrow$).

In Lemmas 1 and 2 it is not assumed that $n = 2$

A topology on a set X is *first countable* if for every $x \in X$ there is a countable collection of open sets containing x such that every open set containing x has a member of the collection as a subset.

Lemma 1. *If a binary relation $\succ$ has a Pareto representation, then the $\succ$ -order topology is first countable.*

Proof. Suppose $\succ$ is a binary relation on X and $v: X \rightarrow \mathbb{R}^n$ is a Pareto representation for $\succ$.

Fix $x \in X$. For $i \in \{1, 2, \dots, n\}$ let Q^i be a countable subset of $B(x)$ such that if $y \succ x$ then there exists $q \in Q^i$ with $v_i(y) \geq v_i(q)$.

For each n -tuple $\langle q^1, q^2, \dots, q^n \rangle \in Q^1 \times Q^2 \times \dots \times Q^n$ choose $z \in X$ such that $v_i(z) = v_i(q^i)$ for all i , if such a z exists. The set Q of all z 's chosen is countable. Form the collection

$$\Omega^W = \{\cap_{i=1}^n W(q^i): q^i \in Q^i \text{ for each } i\} \cup \{W(z): z \in Q\}$$

Since Q is countable and each Q^i is countable, Ω^W is countable.

Now suppose $y \succ x$. For $i \in \{1, 2, \dots, n\}$ choose $q^i \in Q^i$ such that $v_i(y) \geq v_i(q^i)$.

If $v_i(y) > v_i(q^i)$ for some i , then for $z \in \cap_{i=1}^n W(q^i)$, $v(y) > v(z)$ so that $z \in W(y)$. In other words $x \in \cap_{i=1}^n W(q^i) \subseteq W(y)$.

On the other hand, if $v_i(y) = v_i(q^i)$ for all i , then there exists $z \in Q$ such that $v(z) = v(y)$. Then $x \in W(z) \subseteq W(y)$ since $x \in W(y)$ and $W(z) = W(y)$. Also $W(z) \in \Omega^W$ since $z \in Q$.

In summary, if $x \in W(y)$, then $x \in O \subseteq W(y)$ for some $O \subseteq \Omega^W$.

Similarly, there is a countable collection Ω^B of open subsets of X such that $x \in B(y)$ implies $x \in O \subseteq B(y)$ for some $O \in \Omega^B$.

Finally, let

$$\Omega = \{(\cap_{j=1}^k O^j) \cap (\cap_{j=k+1}^l O^j): O^j \in \Omega^W \text{ for } j \in \{1, 2, \dots, k\}, O^j \in \Omega^B \text{ for } j \in \{k+1, \dots, l\}, l \geq 0\}.$$

Then Ω is countable.

If $\Theta \subseteq X$ is open and $x \in \Theta$, by the definition of an order topology there exist $x^1, x^2, \dots, x^k, x^{k+1}, \dots, x^l$ such that $x \in (\cap_{j=1}^k W(x^j)) \cap (\cap_{j=k+1}^l B(x^j)) \subseteq \Theta$. For each $j \in \{1, 2, \dots, k\}$ choose $O^j \in \Omega^W$ such that $x \in O^j \subseteq W(x^j)$. For each $j \in \{k+1, \dots, l\}$

choose $O^j \in \Omega^B$ such that $x \in O^j \subseteq B(x^j)$. Let $O = \cap_{j=1}^l O^j$. Then $x \in O \subseteq \Theta$ and $O \in \Omega$. ■

Suppose $v: X \rightarrow \mathbb{R}^2$ is a Pareto representation for binary relation $\succ$ on X . Construct Pareto representation $V: X \rightarrow \mathbb{R}^2$ for $\succ$ as in Theorem 1 ($\Leftarrow$). Then for $x, y \in X$ $V(x) = V(y)$ if and only if $W(x) = W(y)$ and $B(x) = B(y)$.

Lemma 2. *There exists countable $Q \subseteq X$ such that for $i = 1, 2$ and $x, y \in X$*

$$\begin{aligned} V_i(x) > V_i(y) \text{ implies } V_i(x) > V_i(q) > V_i(y) \\ \text{or } V_i(x) > V_i(q) = V_i(y) \text{ and } V_{-i}(q) \geq V_{-i}(y) \text{ for some } q \in Q \end{aligned} \quad (2)$$

$$\begin{aligned} V_i(x) = V_i(y) \in V_i(Q) \text{ and } V_{-i}(x) > V_{-i}(y) \text{ together imply} \\ V_{-i}(x) > V_{-i}(q) \geq V_{-i}(y) \text{ and } V_i(q) = V_i(y) \text{ for some } q \in Q \end{aligned} \quad (3)$$

Proof. Define $\succ_1^*$ on X by $x \succ_1^* y$ if $V_1(x) > V_1(y)$. Since $\succ_1^*$ is represented by a utility function— V_1 —by Debreu [1964] there exists $S \subseteq X$ such that S is countable and dense in $\succ_1^*$; that is, $V_1(x) > V_1(y)$ implies $V_1(x) > V_1(s) \geq V_1(y)$ for some $s \in S$. Let $\{r^j\} = V_1(S)$. For each j using Debreu (1964) again let Q^j be a countable subset of $V_1^{-1}(r^j)$ such that for $x, y \in V_1^{-1}(r^j)$, $V_2(x) > V_2(y)$ implies $V_2(x) > V_2(q) \geq V_2(y)$ for some $q \in Q^j$ and for $x \in V_1^{-1}(r^j)$, $V_2(q) \geq V_2(x)$ for some $q \in Q^j$. Let $Q^* = S \cup (\cup_j Q^j)$.

If $x, y \in X$ and $V_1(x) > V_1(y)$ then there exists $s \in S \subseteq Q^*$ such that $V_1(x) > V_1(s) \geq V_1(y)$. If $V_1(s) > V_1(y)$, then $V_1(x) > V_1(s) > V_1(y)$. If $V_1(s) = V_1(y) = r^j$, then $V_1(x) > V_1(q) = V_1(y)$ and $V_2(q) \geq V_2(y)$ for some $q \in Q^j \subseteq Q^*$. This establishes that Q^* satisfies (2) for $i = 1$.

If $x, y \in X$, $V_1(x) = V_1(y) \in V_1(Q)$ and $V_2(x) > V_2(y)$ then $V_2(x) > V_2(q) \geq V_2(y)$ for some $q \in Q^j \subseteq Q$ where $V_1(x) = r^j$. Therefore Q^* satisfies (3) for $i = 1$.

Similarly define Q^{**} satisfying (2) and (3) for $i = 2$.

Let $Q = Q^* \cup Q^{**}$ ■

Lemma 3. Suppose $\succ$, X , V and Q are as above. There exists a Pareto representation $f: X \rightarrow \mathfrak{R}^2$ for $\succ$ such that for $x, y \in X$ and $i = 1, 2$

$$f(x) = f(y) \text{ if and only if } V(x) = V(y) \quad (4)$$

$$f_i(x) > f_i(y) \text{ implies } f_i(x) > f_i(q) > f_i(y) \text{ for some } q \in Q \text{ or } f(q) = f(y) \text{ for some } q \in Q \quad (5)$$

for $i = 1, 2$ and $p, q \in Q$

$$f(p) \neq f(q) \text{ implies } f_i(p) \neq f_i(q) \quad (6)$$

Proof. Step 1. Let $P = \{p^j\}$ be a subset of Q such that $V_1(P) = V_1(Q)$ and such that $p^j \neq p^k$ implies $V_1(p^j) \neq V_1(p^k)$. Construct a set $\{I^j\}$ of mutually disjoint, nondegenerate closed intervals in $\mathfrak{R}$ such that $\{I^j\}$ and $\{p^j\}$ have the same index set (the positive integers or a finite initial sequence of the positive integers); for all j, k , $I^j < I^k$ if and only if $V_1(p^j) < V_1(p^k)$, (here $I^j < I^k$ means $\max I^j < \min I^k$); and there is a 1-1 order preserving function $h: \mathfrak{R} \rightarrow \mathfrak{R}$ such that $h(V_1(p^j)) \in V_1(I^j)$ for all j and $h(\mathfrak{R} - V_1(P)) \subseteq \mathfrak{R} - \cup_j I^j$.

Step 2. Fix $p^{j^0} \in \{p^j\}$. Let $X(p^{j^0}) = \{x \in X: V_1(x) = V_1(p^{j^0})\}$. Let g be a 1-1 order preserving function from $\mathfrak{R}$ to I^{j^0} . Define $U: X(p^{j^0}) \rightarrow I^{j^0}$ by $U(x) = g(V_2(x))$. Define U similarly on each $X(p^j)$.

Step 3. Define $f_1: X \rightarrow \mathfrak{R}$ by

$$f_1(x) = \begin{cases} U(x) & \text{if } x \in \cup_j X(p^j) \\ h(V_1(x)) & \text{if } x \in X - \cup_j X(p^j). \end{cases}$$

Step 4. Define $f_2: X \rightarrow \mathfrak{R}$ similarly.

Step 5. Let $f = \langle f_1, f_2 \rangle$.

Clearly $V(x) = V(y)$ if and only if $f(x) = f(y)$; that is, (4) holds.

Now suppose $p, q \in Q$ and $f(p) \neq f(q)$. By (4), $V(p) \neq V(q)$. Therefore either $V_1(p) \neq V_1(q)$, in which case $f_1(p) \neq f_1(q)$, or $V_1(p) = V_1(q)$ and $V_2(p) \neq V_2(q)$ in which case $f_2(p) \neq f_2(q)$. Similarly $f_2(p) \neq f_2(q)$. Therefore (6) holds.

For $x, y \in X$, $x \succ y$ implies $V(x) > V(y)$, which implies $V_1(x) > V_1(y)$ and $V_2(x) > V_2(y)$, or $V_1(x) > V_1(y)$ and $V_2(x) = V_2(y)$, or $V_1(x) = V_1(y)$ and $V_2(x) > V_2(y)$. These three contingencies imply, respectively, $f_1(x) > f_1(y)$ and $f_2(x) > f_2(y)$; $f_1(x) > f_1(y)$ and $f_2(x) \geq f_2(y)$; $f_1(x) \geq f_1(y)$ and $f_2(x) > f_2(y)$. Summarizing, $x \succ y$ implies $f(x) > f(y)$. Also $x \succeq y$ implies $V_1(x) > V_1(y)$ or $V_2(x) > V_2(y)$ which imply respectively $f_1(x) > f_1(y)$; $f_2(x) > f_2(y)$. In other words, $x \succeq y$ implies $f(x) \geq f(y)$. Therefore f is a Pareto representation for $\succ$.

Now to establish (5) suppose $x, y \in X$ and $f_1(x) > f_1(y)$.

Case 1. $V_1(x) > V_1(y)$. By Lemma 2 there exists $q \in Q$ such that either $V_1(x) > V_1(q) > V_1(y)$, which implies $f_1(x) > f_1(q) > f_1(y)$; or $V_1(x) > V_1(q) = V_1(y)$ and $V_2(q) > V_2(y)$, which implies $f_1(x) > f_1(q) > f_1(y)$; or $V_1(x) > V_1(q) = V_1(y)$ and $V_2(q) = V_2(y)$, which by (4) implies $f(q) = f(y)$.

Case 2. $V_1(x) = V_1(y) \in V_1(Q)$ and $V_2(x) > V_2(y)$. Then by Lemma 2 there exists $q \in Q$ such that either $V_2(x) > V_2(q) > V_2(y)$ and $V_1(q) = V_1(y)$ so that $f_1(x) > f_1(q) > f_1(y)$; or $V_2(x) > V_2(q) = V_2(y)$ and $V_1(q) = V_1(y)$ so that by (4) $f(q) = f(y)$.

Similarly, (5) holds for $i = 2$. ■

For $i = 1, 2$ let $\succ_i$ be the binary relation on X defined by $x \succ_i y$ if $f_i(x) > f_i(y)$. Since $\succ_i$ is represented by a utility function, by Debreu [1964] $\succ_i$ is represented by a utility function $w_i: X \rightarrow \mathbb{R}$ such that w_i is continuous as a function from X with the $\succ_i$ -order topology to $\mathbb{R}$. Since f is a Pareto representation for $\succ$ and $w_i(x) > w_i(y)$ if and only if $f_i(x) > f_i(y)$ for all $x, y \in X$, $i = 1, 2$, it follows that $w = \langle w_1, w_2 \rangle$ is a Pareto representation for $\succ$. It remains to show that w_i is continuous as a function from X with the $\succ$ -order topology to $\mathbb{R}$. Without loss of generality, it is enough to show that w_1 is continuous.

By Lemma 1, the $\succ$ -order topology on X is first countable. Therefore by a well-known and easy-to-prove theorem of elementary topology, is enough to show that w_1 is sequentially continuous.

Suppose on the contrary that $\langle y^j \rangle$ is a sequence in X , $x \in X$ and $y^j \rightarrow x$ in

the $\succ$ -order topology on X , but it is not the case that $w_1(y^j) \rightarrow w_1(x)$. Without loss of generality, assume $w_1(y^j) < w_1(x)$ for all j .

Lemma 4. *There exists $\langle x^j \rangle$ in X and $z \in X$ such that $x^j \rightarrow x$ in the $\succ$ -order topology on X and $w_1(x^j) \leq w_1(z) < w_1(x)$ for all j .*

Proof. Since it is not the case that $w_1(y^j) \rightarrow w_1(x)$, there is a subsequence $\langle x^j \rangle$ of $\langle y^j \rangle$ such that if $\langle z^j \rangle$ is a subsequence of $\langle x^j \rangle$ then it is not the case that $w_1(z^j) \rightarrow w_1(x)$.

Now suppose there is no $z \in X$ such that $w_1(x^j) \leq w_1(z) < w_1(x)$ for all j . Then there is a subsequence $\langle z^j \rangle$ of $\langle x^j \rangle$ such that $z^j \rightarrow x$ in the $\succ_1$ -order topology so that by the continuity of w_1 (as a function from X with the $\succ_1$ -order topology to $\mathbb{R}$) $w_1(z^j) \rightarrow w_1(x)$, a contradiction. ■

Notice that for all $i = 1, 2$ and $x, y \in X$ $w_i(x) > w_i(y)$ if and only if $f_i(x) > f_i(y)$, so that Lemma 3 holds with w in place of f .

Lemmas 3 and 4 will now be used to complete the proof of Theorem 2 ($\Leftarrow$).

Case 1. $w_2(z) > w_2(x)$ and $w_2(z) \geq w_2(x^j)$ for all j . By Lemma 4, for all j $w(z) > w(x^j)$ or $w(z) = w(x^j)$. Therefore for all j $x^j \notin \{y \in X: y \sim z \text{ and } B(y) \neq B(z) \text{ or } W(y) \neq W(z)\}$, since if $w(z) > w(x^j)$, then $z \succ x^j$ and if $w(z) = w(x^j)$, then $B(z) = B(x^j)$ and $W(z) = W(x^j)$. But $x \in \{y \in X: y \sim z \text{ and } B(y) \neq B(z) \text{ or } W(y) \neq W(z)\}$ since $w_1(x) > w_1(z)$ and $w_2(x) < w_2(z)$ together imply $x \sim z$, and $w(x) \neq w(z)$ implies (by (4) with w in place of f) $V(x) \neq V(z)$ implies (by the construction of V) $B(x) \neq B(z)$ or $W(x) \neq W(z)$. Also, $\{y \in X: y \sim z \text{ and } B(y) \neq B(z) \text{ or } W(y) \neq W(z)\}$ is open since $\succ$ is weakly order discrete. Therefore it is not the case that $x^j \rightarrow x$ in the $\succ$ -order topology, contradicting the assumption that $x^j \rightarrow x$.

Case 2. $w_2(x^j) > w_2(z) > w_2(x)$ for all j . Suppose $O \subseteq X$ is open and $x \in O$. Then $x \in (\cap_{j=1}^k W(y^j)) \cap (\cap_{j=k+1}^l B(y^j)) \subseteq O$ for some $y^1, y^2, \dots, y^l \in X$, $l \geq k$ nonnegative integers. Since $x^j \rightarrow x$, $x^J \in (\cap_{j=1}^k W(y^j)) \cap (\cap_{j=k+1}^l B(y^j))$ for some J . Then both $w(x)$

and $w(x^j)$ are in $(\cap_{j=1}^k W(w(y^j))) \cap (\cap_{j=k+1}^l B(w(y^j)))$. Since $w_2(x^j) > w_2(z) > w_2(x)$ and $w_1(x^j) \leq w_1(z) < w_1(x)$, $w(z) \in (\cap_{j=1}^k W(w(y^j))) \cap (\cap_{j=k+1}^l B(w(y^j)))$ and therefore $z \in (\cap_{j=1}^k W(y^j)) \cap (\cap_{j=k+1}^l B(y^j)) \subseteq O$. Since $z \in O$ for every open O with $x \in O$, $z^j \rightarrow x$ where $z^j = z$ for every j . The sequence $\langle z^j \rangle$ has $z^j \rightarrow x$, $w_1(z^j) \leq w_1(z) < w_1(x)$, for all j , $w_2(z) > w_2(x)$ and $w_2(z) \geq w_2(z^j)$ for all j . Therefore $\langle z^j \rangle$ satisfies the hypotheses of case 1, which led to a contradiction.

Case 3. $w_2(x) \geq w_2(z)$ and $\{y \in X: w_1(x^j) \leq w_1(y) < w_1(x) \text{ and } w_2(y) > w_2(x)\} = \emptyset$. Since $w_1(x) > w_1(z)$, by the comment that Lemma 3 holds for w , there exists $q \in Q$ such that $w_1(x) > w_1(q) > w_1(z)$ or $w(q) = w(z)$. Also $w_2(x) \geq w_2(q)$ by the assumption of this case. Therefore $w(x) > w(q)$. Therefore $x \in B(q)$.

Since $x \in B(q)$ and $x^j \rightarrow x$, there exists J such that $x^J \in B(q)$. Since $w_1(q) \geq w_1(z) \geq w_1(x^J) \geq w_1(q)$ it follows that $w(q) = w(z)$ and $w_2(x) \geq w_2(x^J) > w_2(q)$.

Using Lemma 3 again, there exists $p \in Q$ such that $w_1(x) > w_1(p) > w_1(x^J)$ or $w(p) = w(x^J)$. Also $w_2(x) \geq w_2(p)$ by the assumption of this case. Therefore $w(x) > w(p)$.

Since $x \in B(p)$, there exists K such that $x^K \in B(p)$. Since $w_1(z) \geq w_1(x^K) \geq w_1(p) \geq w_1(x^J) = w_1(z)$, it follows that $w_1(p) = w_1(q) = w_1(x^J)$. Since $w_1(p) = w_1(x^J)$, $w(p) = w(x^J)$. Therefore $w_2(p) > w_2(q)$. Together, $p, q \in Q$, $w_1(p) = w_1(q)$ and $w_2(p) > w_2(q)$ contradict Lemma 3, (6). ■

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