University of Connecticut

Department of Economics Working Paper Series

Tax Evasion and Financial Repression

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Working Paper 2005-34R

July 2005, revised June 2007

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Abstract

Using a simple overlapping generations framework, calibrated to four Southern European countries, we analyze the relationship between tax evasion, determined endogenously, and financial repression. We show that higher degree of tax evasion within a country, resulting from a higher level of corruption and a lower penalty rate, yields higher degrees of financial repression as a social optimum. However, a higher degree of tax evasion, due to a lower tax rate, reduces the severity of financial restriction.

Journal of Economic Literature Classification: E26, E63

Keywords: Underground Economy; Tax evasion; Macroeconomic Policy.

This paper previously circulated under the title ”Policy Response of Endogenous Tax Evasion”

1 Introduction

Using a simple overlapping generations framework, we analyze the relationship between tax evasion, determined endogenously, and financial repression. We follow Drazen (1989), Bacchetta and Caminal (1992), Haslag and Hein(1995), Espinosa and Yip (1996), Haslag (1998) and Haslag and Koo (1999), in defining financial repression through an obligatory "high" reserve deposit ratio requirement. The study attempts to assay whether there exists a plausible explanation as to why the reserve requirements in some economies are higher than others. Specifically, we analyze whether the "high" reserve requirements are a fall out of a welfare maximizing decision of the government, in an economy characterized by tax evasion.

Note, financial repression can be broadly defined as a set of government legal restrictions, like interest rate ceilings, compulsory credit allocation and high reserve requirements, that generally prevent the financial intermediaries from functioning at their full capacity level. However, given the wave of interest rate deregulation in the 1980s, and removal credit ceiling some years earlier, the major form of financial repression is currently via obligatory reserve requirements.[1] As Espinosa and Yip (1996) points out the concern is not whether financial repression is prevelent but the associated degree to which an economy is repressed, since developed or developing economies both resort to such restrictive policies.

Now the pertinent question here is - Why, if at all, would a government want to repress the financial system ? This seems paradoxical, especially when one takes into account the well documented importance of the financial intermediation process on economic activity, mainly via the

[1] See Demirgüç-Kunt and Detragiache (2001) for further details.

finance-growth nexus.[2] Besides, the fact that "high" cash reserve requirements enhances the size of the implicit tax base and, hence, is lucrative for the government to repress the financial system, an alternative line of thought is derived from the works of Cukierman, Edwards and Tabellini (1992) and Giovannini and De Melo (1993). Both these studies suggested that, countries with an inefficient tax systems would be more oriented towards the repression of the financial sector. Roubini and Sala-i-Martin (1995) addresses this issue in a formal fashion, using an endogenous growth framework. They indicated that, governments subjected to large tax-evasion will "choose to increase seigniorage by repressing the financial sector and increasing the inflation rates."

However, Gupta (2005 and 2006), using a pure-exchange- and a production-economy in an overlapping generations framework, calibrated to Southern European economies, showed that higher tax evasion would cause a benevolent social planner to optimally increase the tax rates, when implicit taxation is also available as a source of revenue. The optimal reserve requirements, however, continued to be at zero, implying the inability of tax evasion to explain financial repression. Similar results have also been obtained by Holman and Neanidis (2006) in the context of an open economy, characterized by both tax evasion and currency substitution.

However, all the above mentioned theoretical analyses, suffer from a serious problem, in the sense, that they treat tax evasion as exogenous. The optimal degree of tax evasion is a behavioral decision made by the agents of the economy, and is likely to be affected by not only the structural parameters of the economy, but also the policy decisions of the government. In such a situation, all these models, essentially suffer from the "Lucas Critique" in some ways. Specifically, all the above studies, look at the optimal policy decisions of the government following an increase in

[2] See Roubini and Sala-i-Martin (1992), and the references cited there in.

the exogenous rate of tax evasion, without specifying what is causing the change in the degree of evasion in the first place. Under such circumstances, the optimal choices made by the government, following an exogenous increase in the degree of tax evasion, is likely to non-optimal, simply because, the degree of tax evasion, following such policy choices, would have changed the actual level of the tax evasion further, once we treat tax evasion as endogenous. What this implies is that once we determine which policy parameters, besides the structural parameters, are affecting the degree of tax evasion, they cannot be available to the government for use to respond optimally to a change in tax evasion. Moreover, once we endogenize the process of tax evasion, it is likely that optimal policy decisions made by the government with regard to the policy instruments available, might vary depending on which factor is driving the change in the degree of evasion.

This paper, thus, extends the work of Roubini and Sala-i-Martin (1995), Gupta (2005 and 2006) and Holman and Neanidis (2006), by, first, providing the microeconomic foundations to the process of tax evasion, along the lines of Atolia (2003), Chen (2003) and Arana (2004), and second, by analyzing how a welfare maximizing social planner will respond to, in terms of its available policy choices, following an increase in the degree of tax evasion, and, in turn, can financial repression, measured by reserve requirements be motivated by tax evasion. To the best of our knowledge, such an attempt to rationalize financial repression based on endogenously determined tax evasion, is the first of its kind.

To validate our analysis, the theoretical model is numerically analyzed by calibrating it to four southern European economies, namely, Greece, Italy, Portugal and Spain. It must, however, be noted that our model economy is a general one, and can be applied to any economy subjected to tax evasion. Our restriction of the applicability to other economies, was plainly due to the unavailability of data regarding the sizes of the underground economy, hence, tax evasion, and production parameters. Moreover, the chosen economies, have had the tradition of high sizes of underground economy and, hence, tax evasion (See Schneider and Klinglmair (2004).). Besides, their reliance on seigniorage, through high inflation rates and reserve requirements, is also well documented in the literature (See Table 1.).

[INSERT TABLE 1]

As suggested at the onset, the paper incorporates endogenous tax evasion in a standard general equilibrium model of overlapping generations. There are two primary assets in the model bank deposits and fiat money. Deposits dominate money in rate of return. An intermediary exists to provide a rudimentary pooling function, accepting deposits to finance the investment needs of the firms, but are subjected to mandatory cash-reserve requirements. There is also an infinitely lived government with two wings: a treasury which finances expenditure by taxing income and setting penalty for tax evasion when caught; and the central bank, which controls the growth rate of the nominal stock of money and the reserve requirements. In such an environment, we deduce the optimal degree of tax evasion, derived from the consumer optimization problem, as function of the parameters and policy variables of the model. The paper is organized as follows: Section 2 lays out the economic environment; Section 3, 4 and 5 respectively, are devoted in defining the monetary competitive equilibrium, discussing the process of calibration, and analyzing the welfare-maximizing choices of policy following an increase in tax evasion, resulting from either policy changes or alteration to structural parameters of the model. Section 7 concludes and lays out the areas of further research.

2 Economic Environment

Time is divided into discrete segments, and is indexed by t = 1, 2,...... There are four theaters of economic activities: (i) each two-period lived overlapping generations household (consumer/worker) is endowed with one unit of labor when young, but the agent retires when old. Thus, at each date t, there are two coexisting generations of young and old. N people are born at each time point $t \geq 1$. At t = 1, there exist N people in the economy, called the initial old, who live for only one period. Hereafter N is normalized to 1. The young inelastically supplies one unit of the labor endowment to earn wage income, a part of the tax-liability is evaded, with evasion being determined endogenously to maximize utility, and the rest is deposited into banks for future consumption. (ii) each infinitely lived producer is endowed with a production technology to manufacture the single final good, using the inelastically supplied labor, physical capital and credit facilitated by the financial intermediaries; (iii) the banks simply converts one period deposit contracts into loans, after meeting the cash reserve requirements. No resources are assumed to be spent in running the banks, and; (iv) there is an infinitely lived government which meets its expenditure by taxing income, setting penalty for tax evasion when caught, and controlling the inflation tax instruments – the money growth rate and the reserve requirements. There is a continuum of each type of economic agents with unit mass.

The sequence of events can be outlined as follows: When young a household works receives pre-paid wages, evades a part of the tax burden and deposits the rest into banks. A bank, after meeting the reserve requirement, provides a loan to a goods producer, which subsequently manufactures the final good and returns the loan with interests. Finally, the banks pay back the deposits with interests to households at the end of the first period and the latter consumes in the second period.

2.1 Consumers

2.1.1 Consumers with Deposit Insurance

Given that the consumers possess an unit of time endowment which is supplied inelastically, and consumes only when old, formally the problem of the consumer can be described as follows: The utility of a consumer born at t depends on real consumption, c_{t+1}, implying that the consumer consumes only when old. The assumptions make computations tractable and is not a bad approximation of the real world (See Hall (1988).). All consumers have the same preferences so that there exists a representative consumer in each generation. The utility function of a consumer born at time t can be written as follows:

$$U_t = u(c_{t+1}) \tag{1}$$

where U is twice differentiable; moreover $u' > 0$ and $u'' < 0$ and $u'(0) = \infty$. The above utility function is maximized subject to the following constraints:

$$\begin{aligned} D_t &\leq q[(1-\beta\tau_t)p_t w_t - \eta(1-\beta)^2 p_t w_t] & (2) \\ &+ (1-q)[(1-\beta\tau_t)p_t w_t - \theta_t\tau_t(1-\beta)p_t w_t - \eta(1-\beta)^2 p_t w_t] & \\ p_{t+1}c_{t+1} &\leq (1+i_{Dt+1})D_t & (3) \end{aligned}$$

where equation (2) is the feasibility (first-period) budget constraint and equation (3) denotes the second period budget constraint for the consumer. Note to be consistent with Chen (2003) and Arana (2004), we start off by assuming that each cohort pool resources and consume at the end of the day. With no aggregate shocks in the economy, we end up having equation (2) which implicitly assumes the existence of some sort of an insurance mechanism that always ensures the consumer a certain amount of deposits, d_t, in our case. Alternatively, we could have assumed the consumer to be risk-neutral. In that case, equation (2) would be the expected value of the deposits obtained. So in some sense we have an observationally equivalent formulation. Our results are, however, independent of whether the consumer is risk-averse or risk neutral, once we assume the existence of an implicit insurance scheme. p_t (p_{t+1}) denotes the money price of the final good at t ($t+1$); D_t is the per-capita nominal deposits; $1-q$ is the probability of getting caught when evading tax; β is the fraction of tax paid; τ_t is the income tax rate at t; θ_t is the penalty imposed, when audited and caught, at t; w_t is the real wage at t, $\eta > 0$, is a cost parameter, and; i_{Dt+1} is the nominal interest rate received on the deposits at $t+1$.

The constraints can be explained as follows: For the potential evader, there are (ex-ante) two possible situations: "success" (i.e., getting away with evasion) and "failure" (i.e., getting discovered and being convicted). If the consumer is found guilty of concealing an amount of income $(1-\beta)p_t w_t$, then he has to pay the amount of the evaded tax liability, $(1-\beta)\tau_t p_t w_t$ and a proportional fine at a rate of $\theta_t > 1$. Notice we have assumed that the household has to incur transaction costs to evade taxes. These basically involve costs of hiring lawyers to avoid/reduce tax burdens, and bribes paid to tax officials and administrators. The transaction costs are incurred in evading taxes are assumed to be increasing in both degree of tax evasion and the wage income of the household. The form $\eta\,(1-\beta)^2\,p_t w_t$ is consistent with our assumptions about the behavior of transaction costs. Note a higher value of η, would imply a less corrupted economy, implying that it is more difficult to evade taxes. We also endogenize the probability of getting caught, $1-q$, by assuming it to be an increasing function of the degree of tax evasion. $1 - q$ takes the following quadratic form:

$$1 - q = (1 - \beta)^2 \tag{4}$$

The second-period budget constraint is self-explanatory suggesting that the consumer when old consumes out of the interest income from deposits – the only source of income, given that he is retired. The household chooses β to maximize his utility from second-period consumption subject to the intertemporal budget constraint given as follows:

$$c_{t+1} \leq (1 + r_{dt+1})[(1 - \beta\tau_t) - \theta_t\tau_t(1 - \beta)^3 - \eta(1 - \beta)^2]w_t \tag{5}$$

where r_{dt+1} is the real interest rate on deposits at period $t + 1$. Note $(1 + r_{dt+1}) = \frac{1+i_{Dt+1}}{1+\pi_{t+1}}$, where $1 + \pi_{t+1} = \frac{p_{t+1}}{p_t}$.

given that the consumer is too small to affect aggregate variables, the first order condition for the consumer would essentially imply, that the agent maximizes the size of the deposits by choosing $beta$. Formally, this is given as follows:

$$\frac{dd_t}{d\beta} = 0 = \frac{d}{d\beta}[(1 - \beta\tau_t) - \theta_t\tau_t(1 - \beta)^3 - \eta(1 - \beta)^2] \tag{6}$$

where $d_t = (\frac{D_t}{p_t})$ is the size of deposits in real terms.

2.1.2 Consumers without Deposit Insurance

Next, we relax the assumption of the existence of an implicit deposit insurance scheme, and reformulate the consumer's problem. Based on the structure, the optimal degree of tax evasion is obtained by solving the following problem:

$$\frac{d}{d\beta}[\{1 - (1 - \beta)^2\}U(c^1_{t+1}) + (1 - \beta)^2U(c^2_{t+1})] = 0 \tag{7}$$

where, c^1_{t+1} and c^2_{t+1} are the consumption levels when the consumer can evade taxes with success and failure, respectively. Mathematically we have the following,

$$c^1_{t+1} = (1 + r_{Dt+1})[(1 - \beta\tau_t) - \eta(1 - \beta)^2]w_t \tag{8}$$

$$c^1_{t+1} = (1 + r_{Dt+1})[(1 - \beta\tau) - \theta_t\tau_t(1 - \beta) - \eta(1 - \beta)^2]w_t \tag{9}$$

Note the definition of all the variables are retained as above. In addition we assume an utility function of the following form, for i = 1, 2:

$$U(c^i_{t+1}) = \frac{(c^i_{t+1})^{(1-\sigma)}}{1 - \sigma} \tag{10}$$

2.2 Financial Intermediaries

At the start of each period the financial intermediaries accept deposits and make their portfolio decision (that is, loans and cash reserves choices) with a goal of maximizing profits. At the end of the period they receive their interest income from the loans made and meets the interest obligations on the deposits. Note the intermediaries are constrained by legal requirements on the choice of their portfolio (that is, reserve requirements), as well as by feasibility. Given such a structure, the intermediaries obtains the optimal choice for L_t by solving the following problem:

$$\max_{L,D} \pi_b = i_{Lt}L_t - i_{Dt}D_t \tag{11}$$

$$s.t. \; : \; \gamma_t D_t + L_t \leqslant D_t \tag{12}$$

where π_b is the profit function for the financial intermediary, and $M_t \geqslant \gamma_t D_t$ defines the legal reserve requirement. M_t is the cash reserves held by the bank; L_t is the loans; i_{Lt} is the interest rate on loans, and; γ_t is the reserve requirement ratio. The reserve requirement ratio is the ratio of required reserves (which must be held in form of currency) to deposits.

To gain some economic intuition of the role of reserve requirements, let us consider the solution of the problem for a typical intermediary. Free entry, drives profits to zero and we have

$$i_{Lt}(1-\gamma_t) - i_{Dt} = 0 \tag{13}$$

Simplifying, in equilibrium, the following condition must hold

$$i_{Lt} = \frac{i_{Dt}}{1-\gamma_t} \tag{14}$$

Reserve requirements thus tend to induce a wedge between the interest rate on savings and lending rates for the financial intermediary.

2.3 Firms

Each firm produces a single final good using a standard neoclassical production function $F(k_t, n_t)$, with k_t and n_t, respectively denoting the capital and labor input at time t. The production technology is assumed to take the Cobb-Douglas form:

$$Y = F(k, n) = k^{\alpha} n^{(1-\alpha)} \tag{15}$$

where $0 < \alpha \ ((1-\alpha)) < 1$, is the elasticity of output with respect to capital (labor). At date t the final good can either be consumed or stored. Next we assume that producers are capable of converting bank loans L_t into fixed capital formation such that $p_t i_{kt} = L_t$, where i_t denotes the investment in physical capital. Notice that the production transformation schedule is linear so that the same technology applies to both capital formation and the production of consumption good and hence both investment and consumption good sell for the same price p. Moreover, we follow Diamond and Yellin (1990) and Chen, Chiang and Wang (2000) in assuming that the goods producer is a residual claimer, i.e., the producer uses up the unsold consumption good in a way which is consistent with lifetime value maximization of the firms. such an assumption regarding ownership avoids the "unnecessary" Arrow-Debreu redistribution from firms to consumers and simultaneously retains the general equilibrium structure.

The representative firm at any point of time t maximizes the discounted stream of profit flows subject to the capital evolution and loan constraint. Formally, the problem of the firm can be outlined as follows

$$\max_{k_{t+1},n_t} \sum_{i=0}^{\infty} \rho^i [p_t k_t^{\alpha} n_t^{(1-\alpha)} - p_t w_t n_t - (1+i_{Lt})L_t] \tag{16}$$

$$k_{t+1} \leqslant (1-\delta_k)k_t + i_{kt} \tag{17}$$

$$p_t i_{kt} = L_t \tag{18}$$

where ρ is the firm owners (constant) discount factor, and δ_k is the (constant) rate of capital depreciation. The firm solves the above problem to determine the demand for labor and investment.

The firm's problem can be written in the following recursive formulation:

$$V(k_t) = \max_{n,k'} [p_t k_t^{\alpha} n_t^{(1-\alpha)} - p_t w_t n_t - p_t(1+i_{Lt})(k_{t+1} - (1-\delta_k)k_t)] + \rho V(k_{t+1}) \tag{19}$$

The upshot of the above dynamic programming problem are the following first order conditions.

$$k_{t+1} \; : \; (1+i_{Lt})p_t = \rho V'(k_{t+1}) \tag{20}$$

$$(n_t) \; : \; (1-\alpha)\left(\frac{k_t}{n_t}\right)^{\alpha} = w_t \tag{21}$$

And the following envelope condition:

$$V'(k_t) = p_t[\alpha \left(\frac{n_t}{k_t}\right)^{(1-\alpha)} + (1 + i_{Lt})(1 - \delta_k)] \tag{22}$$

Optimization, leads to the following efficiency condition, besides (17), for the production firm.

$$(1 + i_{Lt}) = \rho(1 + \pi_{t+1})[\alpha \left(\frac{n_{t+1}}{k_{t+1}}\right)^{(1-\alpha)} + (1 + i_{Lt+1})(1 - \delta_k)] \tag{23}$$

Equation (19) provides the condition for the optimal investment decision of the firm. The firm compares the cost of increasing investment in the current period with the future stream of benefit generated from the extra capital invested in the current period. And equation (17) simply states that the firm hires labor up to the point where the marginal product of labor equates the real wage.

2.4 Government

As discussed above we have an infinitely lived government with two wings: the treasury and the central bank. The government finances its expenditure $p_t g_t$ through taxation, penalty on consumers when caught evading and the inflation tax (seigniorage). Formally the government budget constraint can be written as follows:

$$p_t g_t = \beta \tau_t p_t w_t + (1 - q)\theta_t (1 - \beta)\tau_t p_t w_t + (M_t - M_{t-1}) \tag{24}$$

Note throughout the analysis we will assume that money growth is dictated by a rule, $M_t = (1 + \mu_t)M_{t-1}$, where μ is the rate of growth of money. Using, $M_t = \gamma_t D_t$, the government budget constraint in real terms can be rewritten as

$$g_t = [\beta + (1 - q)\theta_t(1 - \beta)]\tau_t w_t + \gamma_t d_t(1 - \frac{1}{1 + \mu_t}) \tag{25}$$

Skinner and Slemrod (1985) points out that the administrative costs of penalties is usually quite minor, and, hence, for simplicity we ignore them from the government budget constraint. However, the costs involved in auditing the households have been ignored due to the unavailability of information on such costs for our sample of countries. But adding an extra dimension of cost, merely implies an increase in the public expenditures and an inflated level of the policy parameters, without qualitatively changing our results.

3 Equilibrium

A valid perfect-foresight, competitive equilibrium[3] for this economy is a sequence of prices $\{p_t, i_{Dt}, i_{Lt}\}_{t=0}^{\infty}$, allocations $\{c_{t+1}[c_{t+1}^1, C_{t+1}^2], n_t, i_{kt}\}_{t=0}^{\infty}$, stocks of financial assets $\{m_t, d_t\}_{t=0}^{\infty}$, and policy variables $\{\gamma_t, \mu_t, \tau_t, \theta_t, g_t\}_{t=0}^{\infty}$ such that:

- Taking, τ_t, g_t, θ_t, γ_t, μ_t, p_t,r_{Dt+1} and w_t, the consumer optimally chooses β such that (6) [7] holds;
- Banks maximize profits, taking, i_{Lt}, i_{Dt}, and γ_t as given and such that (14) holds;
- The real allocations solve the firm's date–t profit maximization problem, given prices and policy variables, such that (21) and (23) holds;
- The money market equilibrium conditions: $m_t = \gamma_t d_t$ is satisfied for all $t \geqslant 0$;
- The loanable funds market equilibrium condition: $p_t i_{kt} = (1 - \gamma_t)D_t$ where the total supply of loans $L_t = (1 - \gamma_t)D_t$ is satisfied for all $t \geqslant 0$;

[3]Terms within the square brackets correspond to the case when the consumer does not have access to implicit deposit insurance.

- The goods market equilibrium condition require: $c_t + i_{kt} + g_t = k_t^{\alpha} n_t^{(1-\alpha)}[1-\eta(1-\beta)^2(1-\alpha)]$ is satisfied for all $t \geqslant 0$;
- The labor market equilibrium condition: $(n_t)^d$ =1 for all $t \geqslant 0$;
- The government budget is balanced on a period-by-period basis;
- d_t, r_{dt}, r_{Lt}, and p_t must be positive at all dates.

4 Solving the Model and the Optimal Degree of Tax Evasion

Using the equilibrium conditions, realizing that there is no growth in the model, and allowing the government to follow time invariant policy rules, which means the reserve–ratio, γ_t, the money supply growth–rate, μ_t, the tax–rate, τ_t, and the penalty, θ_t, are constant over time, we have the following set of equations:

$$1 + r_d = (1 + r_l)(1 - \gamma) + \frac{\gamma}{1 + \pi} \tag{26}$$

$$w = (1 - \alpha)k^{\alpha} \tag{27}$$

$$(1 + r_l) = \frac{\rho \alpha k^{(\alpha-1)}}{1 - \rho(1 + \pi)(1 - \delta_k)} \tag{28}$$

$$(1 - \gamma)[(1 - \beta\tau) - \theta\tau(1 - \beta)^3 - \eta(1 - \beta)^2]w = \delta_k k \tag{29}$$

where r_l is the real interest rate on loans. Note equation (26) corresponds to the zero-profit condition in the banking sector, written in real-terms. While, (27) and (28) is obtained from the first-order conditions of the firm's profit-maximization problem. Finally, (29) corresponds to the loanable funds equilibrium, which ensures that total investment needs are financed via loans. Using equations (27) and (29), we can, now, solve for k, in terms of the policy variables, the production parameter and β, and can be formally specified as follows:

$$k = \left(\frac{(1-\gamma)(1-\alpha)[(1-\beta\tau)-\theta\tau(1-\beta)^3-\eta(1-\beta)^2]}{\delta_k}\right)^{\frac{1}{1-\alpha}} \tag{30}$$

Realizing that $\mu = \pi$, from the money market equilibrium condition, the gross real interest rate on deposits $(1+r_d)$, and real wage w are given by the following expressions:

$$\begin{aligned}(1+r_d) &= \left[\frac{\rho\alpha\delta_k}{(1-\alpha)(1-\rho(1+\mu)(1-\delta_k))}\right]\left[\frac{1}{[(1-\beta\tau)-\theta\tau(1-\beta)^3-\eta(1-\beta)^2]}\right] \qquad (31)\\ &+ \frac{\gamma}{1+\mu}\\ w &= (1-\alpha)\left\{(1-\alpha)(1-\gamma)\frac{[(1-\beta\tau)-\theta\tau(1-\beta)^3-\eta(1-\beta)^2]}{\delta_k}\right\}^{\frac{\alpha}{(1-\alpha)}} \qquad (32)\end{aligned}$$

where $1+r_l = \left(\frac{\rho\alpha\delta_k}{(1-\gamma)(1-\alpha)(1-\rho(1+\mu)(1-\delta_k))}\right)\left(\frac{1}{[(1-\beta\tau)-\theta\tau(1-\beta)^3-\eta(1-\beta)^2]}\right)$ So from equations (30), (31) and (32) we can see that once we solve for the optimal degree of tax evasion, we can obtain the solutions for the per-capita capital stock, real interest rates and the real-wage rate in steady-state, in terms of the parameters and policy variables of the model. Further, c_{t+1}, c^1_{t+1} and c^2_{t+1} are also determined once we find the optimal value of β.

The optimal degree of tax-evasion, on the other hand, is obtained from equation (6), when there exists a deposit insurance scheme, and from equations (7), (8), (9) and (10) when no such scheme is available to the consumers.

Based on equation (6), we obtain two solutions for the optimal fraction of reported income, β^*, as follows:

$$\beta^* = \frac{\eta + 3\,\theta\,\tau - \sqrt{\eta^2 + 3\,\theta\,\tau^2}}{3\,\theta\,\tau} \tag{33}$$

$$\beta^* = \frac{\eta + 3\,\theta\,\tau + \sqrt{\eta^2 + 3\,\theta\,\tau^2}}{3\,\theta\,\tau} \tag{34}$$

But for the case when agents cannot insure their deposits, we cannot obtain solutions of β^*, unless we specify the value of σ, the degree of risk aversion. Choosing $\sigma = 2$,[4] we end up having six roots of β^*, obtained from the solution of equation (7), given (8), (9) and (10). To save space, all the roots have been reported in the appendix.[5]

Given, the complexity of the expressions, the choice of parameter values becomes paramount for not only choosing a valid root of β^*, i.e., the root that lies between 0 and 1, but also for analyzing the movements in the optimal degree of tax evasion, following changes in public policy. However, an important result that emerges from from all the solutions of β^* is that the optimal degree of tax evasion is independent of the monetary policy parameters of the model, namely γ and μ. This is simply because the real interest rate on deposits does not affect the choice of β^*. This is true even for agents who does not have access to deposit insurance scheme. Hence, when carrying out the welfare analysis, to check whether financial repression and tax evasion are positively related, the only available set of policy parameters for the government would be the reserve requirements and the money growth rate. However, given that tax evasion is affected by η, τ and θ, we analyze the optimal choices of γ and μ, following a change in tax evasion from all the three sources. The results would tell us not only if, but more importantly when, tax evasion and reserve requirements, and, hence, financial repression are positively correlated. With the welfare maximizing problem being a non-linear one, the solution of the same requires numerical values of the parameters of the model, which, in turn, also makes the role of calibration highly essential.

[4]Note alternative values of σ, such as 1 and $\frac{1}{2}$, would imply that the model cannot be solved in an algebraic fashion. Hence, we choose a value of 2.

[5]Note all the calculations were made in Mathematica 5.0.

5 Calibration

In this section, we attribute values to the parameters of our benchmark model using a combination of figures from previous studies and facts about the economic experience for our sample economies between 1980 and 1998.

We follow the standard real business cycle literature in using steady–state conditions to establish parameter values observed in the data. Some parameters are calibrated using country–specific data, while others, without sufficient country–specific evidence over a long period, correspond to prevailing values from the literature. This section reveals the general procedures used. The calibrated parameters are reported in Table 1. Note unless otherwise stated, the source for all data is the IMF – International Financial Statistics (IFS).

A first set of parameter values is given by numbers usually found in the literature. These are:

- σ: the degree of risk aversion, as stated above, is set to 2;
- $(1-\alpha)$: since the production function is Cobb-Douglas, this corresponds to the share of labor in income. $(1-\alpha)$ for Spain, Italy and Greece is derived from Zimmermann (1994) and the value for Portugal is obtained from Correia, Neves and Rebelo (1995). The values lie between 53.0 percent (Portugal) and 62.7 percent (Spain);
- δ_k: the depreciation rate of physical capital for Spain, Italy and Greece is derived from Zimmermann (1997) and the value for Portugal is obtained from Correia, Neves and Rebelo (1995). The values lie between 3.2 percent (Greece) and 5.2 percent (Italy);
- UGE: The parameter measures the size of the underground economy as a percentage of GDP. The values are obtained from Schneider and Klinglmair (2004) and lies between 23.1

percent (Spain and Portugal) to 29.0 percent (Greece);

- θ: the penalty imposed by the government when the consumer is caught evading is obtained from Chen (2003) and is set to 1.5 for all countries.

A second set of parameters is determined individually for each country. Here, we use averages over the whole sample period to find values that do not depend on the current business cycle. These parameters are:

- π: the annual rate of inflation lies between 7.52 percent (Spain) and 15.16 percent (Greece);
- γ: the annual reserve–deposit ratio lies between 13.7 percent (Italy) and 23.5 percent (Greece);
- τ: the tax rate, calculated as the ratio of tax–receipts to GDP, lies between 22.74 percent (Greece) and 36.25 percent (Italy);
- i_{Lt}: the nominal interest rate on loans lies between 12.89 percent (Spain) and 22.96 percent (Greece);
- ϕ: the ratio of government consumption expenditure to GDP is obtained from the data, for individual countries. The country specific value lies between 14.55 percent (Greece) and 18.79 percent (Italy).

The following set of parameters are calibrated from the steady-state equations of the model:

- μ: the money growth rate is set equal to the rate of inflation, given the money market equilibrium condition. The annual rate of money growth rate hence, lies between 7.52 percent (Spain) and 15.16 percent (Greece);

- β: the fraction of reported income is determined by the following method:

$$\frac{TE}{Y} = UGE \times \tau \tag{35}$$

where $\frac{TE}{Y}$ is tax evasion as a percentage of GDP.

We have assumed that the effective average tax rate is the same in the official and the underground economy. Given that

$$(1-\beta) = \frac{\frac{TE}{Y}}{[\frac{TE}{Y} + \tau]} \tag{36}$$

We consider this exogenously evaluated value of the reported income parameter as the steady-state value. Note the value hinges critically on the size of the underground economy as a percentage of the GDP. The value of β lies between 0.775 (Greece) and 0.810 (Spain and Portugal), implying that 22.5 percent of the taxes are evaded in Greece and the figure in Spain and Portugal corresponds to a tax evasion of 19.0 percent.

- η: the cost parameter measuring the resources spent by the households to reduce their tax burden is calibrated to ensure that the optimal degree of tax evasion matches the calculation of $1 - \beta^*$, obtained above. The value lies between 0.3902 (Greece) to 0.6724 (Italy), when there exists a deposit insurance scheme. While, when the agents cannot insure their deposits, the value lies between 0.3751 (Greece) and 0.6270 (Italy). Note that these values are obtained based on equation (33) – the negative root in the case where there exists deposit insurance, and from the second root out of the six, when agents have no access to deposit-insurance. For the other roots, we could not obtain fractional values[6] of η, required to ensure

[6]Either none or imaginary values were obtained for η from the other roots.

that the degree of tax evasion corresponds to our calculations made above, using the size of the underground economy. As a result, based on our choice of η, we eliminate the positive root and the other five roots, when the agents have access to a deposit insurance scheme and when they do not, respectively.

- ρ: the discount factor of the firms is solved to ensure that equation (31) holds. The value ranges between 0.8713 (Greece) to 0.9396 (Spain), when agents are risk-neutral or there exists a deposit insurance scheme, and between 0.8714 (Greece) to 0.9396 (Spain), when agents cannot insure their deposits.

[INSERT TABLE 2]

6 Welfare-Maximizing Monetary Policy in the Presence of Tax Evasion

In this section, we analyze whether higher degree of tax evasion would result in an increase in the degree of financial repression within a specific country. In our case, this implies an increase in γ following a decrease in β^*. For this purpose, we study the behavior of a social planner who maximizes the utility of all consumers, by choosing γ and $\mu = \pi$, subject to the set of inequality constraints: $0 \leq \gamma \leq 1$, $\mu \geq 0$ and the government budget constraint, equation (25), evaluated at the steady state, following changes in β^*. Given that the consumers consume only in the second period, the specification of the utility function required to define the social welfare function is redundant in the case when there exists an implicit deposit insurance scheme. The social planner then, simply maximizes the return on deposits of the agents. However, when agents does not have deposit-insurance, the social planner maximizes $[-\{1-(1-\beta)^2\}\frac{1}{(c^1_{t+1})} - (1-\beta)^2\frac{1}{(c^2_{t+1})}]$, given that σ = 2.

First, we derive the optimal values of γ and μ for the steady-state values of β^* reported in Table 2, and then analyze the movements of the optimal values of monetary policy parameters, following an one percent increase in β^* solely due to a change in either η or τ or θ, as three separate cases. For this purpose, the values of η, τ and θ were re-calibrated, and have been reported in Table 3, under the two alternative scenarios regarding the availability of a deposit insurance scheme.

[INSERT TABLE 3]

This exercise also provided us the opportunity to check whether our model can replicate the behavior of the reported income, following a change in the tax and penalty rates, as observed in recent studies, namely Atolia (2003), Chen (2003) and Arana (2004), dealing with endogenous tax evasion. Figures 1 through 4 and 5 through 8 captures the movement in the optimal fraction of reported income for a change in the tax rate and penalty rate, respectively, under alternative specifications regarding the availability of deposit insurance. Defining β_1 as the path of reported income when the economy is characterized by a deposit insurance scheme and β_2 as the same when agents cannot insure their deposits, we observe that the optimal fraction of reported income, for all four economies, is negatively (positively) related to the tax rate (penalty rate). Moreover, the result holds irrespective of whether the agents are risk-neutral or risk-averse. Understandably, we observe that reported income falls at a slower rate in the case of an agent unable to insure deposits when compared to the case of an agent with deposit insurance. On the other hand, as expected, higher penalty rates causes an insured agent to increase the reported income at a slower rate in comparison to an uninsured agent. Note in the figures we vary the tax rate between the steady-state values of the tax rate of each country, reported in Table 2, and 50 percent, while, the penalty rates are doubled from 1.5 to 3.0. Table 3 also vindicates our findings regarding the movements of reported income following changes to the tax and the penalty rates. Moreover, given that η is the fraction of resources that the consumers need to spend to evade taxes, one percent increase in the reported income, *ceteris paribus* can only be possible due to an increase in the value of η.

[INSERT FIGURES 1 THROUGH 8]

The optimal values of the monetary policy variables, corresponding to $\beta = \beta^*$ and $\beta^* + 0.01$, arising solely due to a change in either η or τ or θ, under alternative assumptions regarding the availability of deposit insurance, are reported in Table 4.

[INSERT TABLE 4]

Under the assumption of deposit insurance, Columns 2 and 3 in Table 4 reports the optimal values of the reserve requirements (γ^*) and the money growth rate (μ^*) for the four countries, given the steady-state values of the reported income (β^*), while columns 4 and 5, 6 and 7, and 8 and 9, studies the movements of γ^* and μ^* respectively, corresponding to one percent increase in the reported income, following an increase in η, an increase in θ and a decrease in τ, respectively. While, Columns 10 to 17, does the same, when consumers cannot insure their deposits. The following inferences can be made based on the results reported in Table 4:

- When agents have access to deposit insurance, an one percent increase in the reported income, emerging from an increase in η and θ, causes the optimal value of the reserve requirement (γ^*) to fall, but the optimal money growth rate (μ^*)[7] to increase, for all the countries.

[7]The optimal money growth rate obtained is exceptionally high, when compared to the actual data in Table 2. However, the results are in line with Freeman (1987). For further details, see Bhattacharya and Haslag (2001).

However, when the increase in the reported income is due to a fall in the tax rate, γ^* rises and μ^* falls;

- In the scenario, when agents cannot insure their deposits, the movements of the optimal reserve requirement (γ^*) and the optimal money growth rate (μ^*), for all the countries, corresponding to one percent increase in β^*, following an increase in η, an increase in θ and a decrease in τ, are same as above, i.e., in the case when agents can insure;
- However, higher degree of tax evasion does not imply higher level of financial restrictions, when we compare across countries, irrespective of whether deposit insurance is available or not available.

So, in summary, we can conclude that irrespective of whether agents can or cannot insure their deposits, reserve requirement and, hence, financial repression is positively correlated with degrees of tax evasion, only when the change in the reported income results from a change in the fraction of resources spent to evade taxes or the penalty rate. Changes in the degree of tax evasion, following a change in the tax rate, however reverses the sign of the correlation between tax evasion and financial repression. Recall, Roubini and Sala-i-Martin (1995) pointed out that governments subjected to large tax-evasion will "choose to increase seigniorage by repressing the financial sector and increasing the inflation rates." In our case though, the first half of the result holds if the increase in the degree of tax evasion results from a lower penalty rate or higher level of corruption, i.e., lesser fraction of resources is needed to be spent to evade tax. While, the second half of the result, dealing with higher inflation rate, holds only if the increase in the degree of tax evasion results from fall in the tax rate. Hence, our analysis points out the importance of modeling tax evasion as an endogenous decision, and the asymmetries that arises in policy conclusions depending upon alternative factors causing a change in the degree of tax evasion.

7 Conclusion and Areas of Further Research

Using a simple overlapping generations framework, we analyze the relationship between tax evasion, determined endogenously, and financial repression. Following the broad literature, we define financial repression through an obligatory "high" reserve deposit ratio requirement. The study attempts to assay whether there exists a plausible explanation as to why the reserve requirements in some economies are higher than others. Specifically, we analyze whether the "high" reserve requirements are a fall out of a welfare maximizing decision of the government, in an economy characterized by tax evasion.

When numerically analyzed for four southern European countries, the following conclusions could be drawn: (i) Increases (decreases) in the penalty rates of evading taxes would induce consumers to report greater (smaller) fraction of their income, while, increases (decreases) in the income-tax rates would cause them to evade greater (lesser) fraction of their income; (ii) Higher degree of tax evasion within a country, resulting from lower penalty rates and higher corruption, produces socially optimum higher degrees of financial repression, i.e., a higher value of the reserve requirement. However, higher degrees of tax evasion, due to lower tax rates, tends to reduce the optimal degree of financial repression; (iii) Higher fraction of reported income, resulting from lower level of corruption or higher penalty rates, causes the government to inflate the economy at a higher rate. Inflation, though, tends to fall, when an increase in the fraction of reported income originates from a fall in the tax rate; and , (iv) Results (i) through (iii) continues to hold irrespective of whether the consumers can or cannot insure their deposits.

So in summary, from a policy perspective, the model suggests that, the best way to reduce tax evasion is by increasing the penalty rates, in cases taxes cannot be reduced due to budgetary pressures. And, an increase in the degree of evasion within the country, resulting from lower penalty rates or higher corruption, should be followed by an increase in the reserve requirements and a decline in the money growth rate as part of a welfare maximizing behavior of the consolidated government. However, higher tax evasion due to lower tax rates, causes the growth rate of money supply and the reserve requirement to move in the opposite direction. Our paper, thus, concludes that there exists asymmetries in optimal monetary policy decisions, depending on what is causing a change in the degree of tax evasion. More importantly though, tax evasion and financial repression are positively correlated, if and only if, the change in the former results from an alteration in the penalty rate or the level of corruption.

An immediate extension of the current model would be repeat the analysis in monetary endogenous growth models, where the endogeneity of the growth process not only originates due to externalities in the production function as in Romer (1986), but also due to productive public expenditure as in Barro (1990).

Acknowledgements This is a revised version of the sixth chapter of my dissertation at the University of Connecticut. I am particularly grateful to my advisors Christian Zimmermann and Dhammika Dharmapala, and David VanHoose and two anonymous referees for many helpful comments and discussions. All remaining errors are mine. **Selected References**

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Table 1: Seigniorage Revenue in some European Economies (1980-2002)

Countries	Seigniorage	Reserves/	Annual	Tax
Countries	(percentages of	Deposits	Inflation	Evasion
Countries	revenue)	(percentage)	rate	(percentage)
Spain	6.4[a]	12.0	6.8	18.0
Greece	19.4[b]	22.9	14.9	22.0
Italy	3.7[b]	11.7	7.5	21.0
Portugal	6.0[c]	17.5	12.2	18.0

Source: IFS – IMF International Financial Statistics.

Seigniorage has been calculated from lines 14a and 14c.

[a]: Excludes the year 2002.

[b]:Excludes the years 1998-2002.

[c]: Excludes the years 1999-2002.

Table 2: Calibration of Parameters

	$(1-\alpha)$	δ_k	UGE	$\pi = \mu$	γ	τ	i_L	ρ		η		β	ϕ
								DI	NDI	DI	NDI		
Spain	0.627	0.05	23.1	7.52	14.1	25.53	12.89	0.9396	0.9396	0.5627	0.5482	0.810	16.66
Italy	0.617	0.052	27.3	8.58	13.7	36.25	15.02	0.9248	0.9249	0.6724	0.6270	0.786	18.79
Greece	0.598	0.032	29.0	15.16	23.5	22.74	22.96	0.8713	0.8714	0.3902	0.3751	0.775	14.55
Portugal	0.530	0.05	23.1	13.04	19.8	27.73	19.09	0.8743	0.8743	0.6112	0.5934	0.810	16.59

Notes: (i) Parameters defined as above.

(ii) DI: Deposit-Insurance; NDI: No Deposit Insurance.

Table 3: Re-calibration of the Tax Evasion Parameters

	DI ($\beta^* + .01$)			NDI ($\beta^* + .01$)		
	η	θ	τ	η	θ	τ
Spain	0.6058	2.1248	0.2371	0.5927	1.9895	0.2356
Italy	0.7221	1.9480	0.3376	0.6807	1.7982	0.3310
Greece	0.4188	1.8904	0.2119	0.4049	1.8134	0.2099
Portugal	0.6580	2.1247	0.2576	0.6419	1.9760	.2556

Notes: (i) Parameters defined as above.

(ii) DI: Deposit-Insurance; NDI: No Deposit Insurance.

Table 4: Optimal Monetary Policy Parameters

	DI								NDI							
	β^*		β^*_η		β^*_θ		β^*_τ		β^*		β^*_η		β^*_θ		β^*_τ	
	γ^*	μ^*	γ^*	μ^*	γ^*	μ^*	γ^*	μ^*	γ^*	μ^*	γ^*	μ^*	γ^*	μ^*	γ^*	μ^*
Spain	0.2206	0.4955	0.2049	0.5236	0.1978	0.5374	0.3103	0.3965	0.0785	1.3223	0.0783	1.3319	0.0776	1.3546	0.0797	1.2372
Italy	0.0210	447260	0.0168	4.9559×10^8	0.0140	3.8049×10^9	0.0463	39.7707	0.1058	1.1620	0.1047	1.1902	0.1046	1.1884	0.1077	1.0646
Greece	0.2438	0.4790	0.2344	0.4881	0.2291	0.4937	0.2982	0.4385	0.0593	1.2283	0.0587	1.2531	0.0589	1.2430	0.4329	0.2652
Portugal	0.2505	0.8352	0.2395	0.8655	0.2348	0.8799	0.3118	0.7134	0.1337	1.2490	0.1313	1.2912	0.1330	1.2626	0.1349	1.1867

Notes: (i) See the notes to Table 3.

(ii) β^*_i, $i = \eta$, θ and τ, measures a one percent increase in β^*_i for a change in i.

Figure 1: Effect of Tax Rate (Spain)

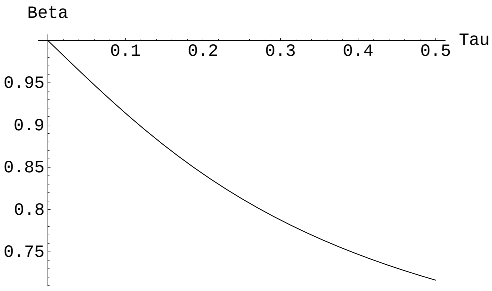


Figure 2: Effect of Tax Rate (Italy)

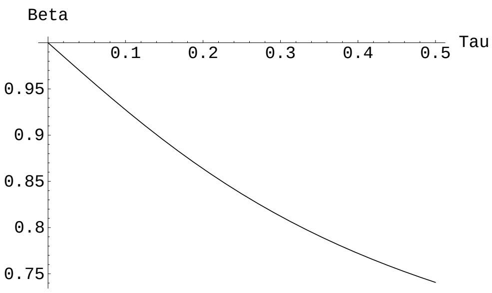


Figure 3: Effect of Tax Rate (Greece)

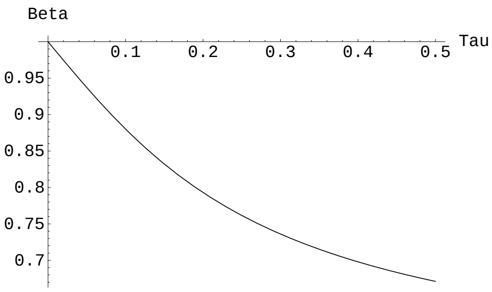


Figure 4: Effect of Tax Rate (Portugal)

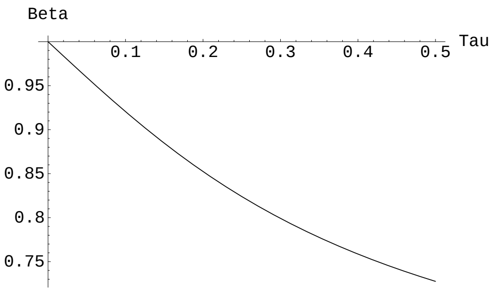


Figure 5: Effect of Penalty Rate (Spain)

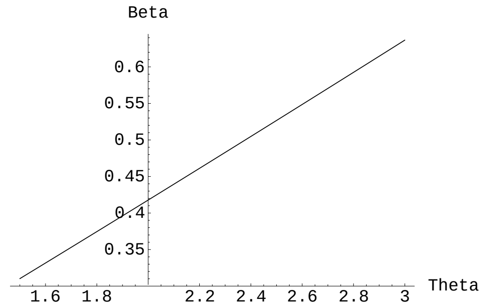


Figure 6: Effect of Penalty Rate (Italy)

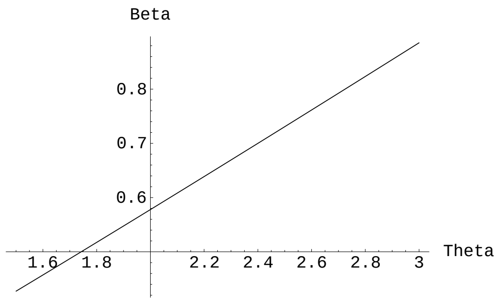


Figure 7: Effect of Penalty Rate (Greece)

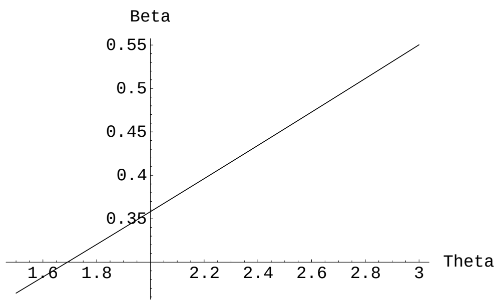
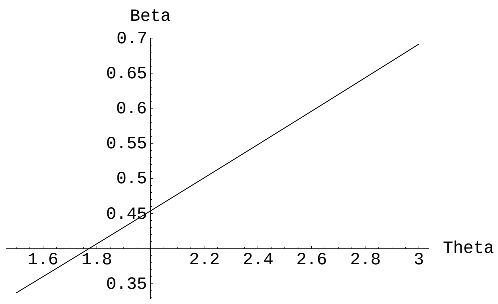


Figure 8: Effect of Penalty Rate (Portugal)



APPENDIX

Six Roots of the Optimal Degree of Tax Evasion when the Consumer does not have Access to Deposit Insurance:

$$
\begin{aligned}
&\{\beta \to \\
&\quad \text{Root}\big[-2\eta + 4\eta^2 - 2\eta^3 + \tau - 2\eta\tau + \eta^2\tau - 3\theta\tau + 6\eta\theta\tau - \\
&\qquad 3\eta^2\theta\tau + 2\theta^2\tau^2 - 2\eta\theta^2\tau^2 + 2\eta\,\#1 - 12\eta^2\,\#1 + \\
&\qquad 10\eta^3\,\#1 + 8\eta\tau\,\#1 - 8\eta^2\tau\,\#1 + 6\theta\tau\,\#1 - 16\eta\theta\tau\,\#1 + \\
&\qquad 10\eta^2\theta\tau\,\#1 - 2\tau^2\,\#1 + 2\eta\tau^2\,\#1 + 2\theta\tau^2\,\#1 - 2\eta\theta\tau^2\,\#1 - \\
&\qquad 6\theta^2\tau^2\,\#1 + 6\eta\theta^2\tau^2\,\#1 + 12\eta^2\,\#1^2 - 20\eta^3\,\#1^2 - \\
&\qquad 6\eta\tau\,\#1^2 + 18\eta^2\tau\,\#1^2 - 3\theta\tau\,\#1^2 + 16\eta\theta\tau\,\#1^2 - \\
&\qquad 9\eta^2\theta\tau\,\#1^2 - 6\eta\tau^2\,\#1^2 - 6\theta\tau^2\,\#1^2 + 2\eta\theta\tau^2\,\#1^2 + \\
&\qquad 6\theta^2\tau^2\,\#1^2 - 6\eta\theta^2\tau^2\,\#1^2 + \tau^3\,\#1^2 + \theta\tau^3\,\#1^2 + \theta^2\tau^3\,\#1^2 - \\
&\qquad 4\eta^2\,\#1^3 + 20\eta^3\,\#1^3 - 16\eta^2\tau\,\#1^3 - 8\eta\theta\tau\,\#1^3 - \\
&\qquad 4\eta^2\theta\tau\,\#1^3 + 4\eta\tau^2\,\#1^3 + 4\theta\tau^2\,\#1^3 + 2\eta\theta\tau^2\,\#1^3 - \\
&\qquad 2\theta^2\tau^2\,\#1^3 + 2\eta\theta^2\tau^2\,\#1^3 - 2\theta^2\tau^3\,\#1^3 - 10\eta^3\,\#1^4 + \\
&\qquad 5\eta^2\tau\,\#1^4 + 2\eta\theta\tau\,\#1^4 + 11\eta^2\theta\tau\,\#1^4 - 2\eta\theta\tau^2\,\#1^4 - \\
&\qquad \theta\tau^3\,\#1^4 + \theta^2\tau^3\,\#1^4 + 2\eta^3\,\#1^5 - 6\eta^2\theta\tau\,\#1^5 + \eta^2\theta\tau\,\#1^6\ \&,\ 1\big]\}
\end{aligned}
$$

$$
\begin{aligned}
&\{\beta \to \\
&\quad \text{Root}\big[-2\eta + 4\eta^2 - 2\eta^3 + \tau - 2\eta\tau + \eta^2\tau - 3\theta\tau + 6\eta\theta\tau - \\
&\qquad 3\eta^2\theta\tau + 2\theta^2\tau^2 - 2\eta\theta^2\tau^2 + 2\eta\,\#1 - 12\eta^2\,\#1 + \\
&\qquad 10\eta^3\,\#1 + 8\eta\tau\,\#1 - 8\eta^2\tau\,\#1 + 6\theta\tau\,\#1 - 16\eta\theta\tau\,\#1 + \\
&\qquad 10\eta^2\theta\tau\,\#1 - 2\tau^2\,\#1 + 2\eta\tau^2\,\#1 + 2\theta\tau^2\,\#1 - 2\eta\theta\tau^2\,\#1 - \\
&\qquad 6\theta^2\tau^2\,\#1 + 6\eta\theta^2\tau^2\,\#1 + 12\eta^2\,\#1^2 - 20\eta^3\,\#1^2 - \\
&\qquad 6\eta\tau\,\#1^2 + 18\eta^2\tau\,\#1^2 - 3\theta\tau\,\#1^2 + 16\eta\theta\tau\,\#1^2 - \\
&\qquad 9\eta^2\theta\tau\,\#1^2 - 6\eta\tau^2\,\#1^2 - 6\theta\tau^2\,\#1^2 + 2\eta\theta\tau^2\,\#1^2 + \\
&\qquad 6\theta^2\tau^2\,\#1^2 - 6\eta\theta^2\tau^2\,\#1^2 + \tau^3\,\#1^2 + \theta\tau^3\,\#1^2 + \theta^2\tau^3\,\#1^2 - \\
&\qquad 4\eta^2\,\#1^3 + 20\eta^3\,\#1^3 - 16\eta^2\tau\,\#1^3 - 8\eta\theta\tau\,\#1^3 - \\
&\qquad 4\eta^2\theta\tau\,\#1^3 + 4\eta\tau^2\,\#1^3 + 4\theta\tau^2\,\#1^3 + 2\eta\theta\tau^2\,\#1^3 - \\
&\qquad 2\theta^2\tau^2\,\#1^3 + 2\eta\theta^2\tau^2\,\#1^3 - 2\theta^2\tau^3\,\#1^3 - 10\eta^3\,\#1^4 + \\
&\qquad 5\eta^2\tau\,\#1^4 + 2\eta\theta\tau\,\#1^4 + 11\eta^2\theta\tau\,\#1^4 - 2\eta\theta\tau^2\,\#1^4 - \\
&\qquad \theta\tau^3\,\#1^4 + \theta^2\tau^3\,\#1^4 + 2\eta^3\,\#1^5 - 6\eta^2\theta\tau\,\#1^5 + \eta^2\theta\tau\,\#1^6\ \&,\ 2\big]\}
\end{aligned}
$$

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\begin{aligned}
&\{\beta \to \\
&\quad \mathrm{Root}\big[-2\eta + 4\eta^2 - 2\eta^3 + \tau - 2\eta\tau + \eta^2\tau - 3\theta\tau + 6\eta\theta\tau - \\
&\qquad 3\eta^2\theta\tau + 2\theta^2\tau^2 - 2\eta\theta^2\tau^2 + 2\eta\,\#1 - 12\eta^2\,\#1 + \\
&\qquad 10\eta^3\,\#1 + 8\eta\tau\,\#1 - 8\eta^2\tau\,\#1 + 6\theta\tau\,\#1 - 16\eta\theta\tau\,\#1 + \\
&\qquad 10\eta^2\theta\tau\,\#1 - 2\tau^2\,\#1 + 2\eta\tau^2\,\#1 + 2\theta\tau^2\,\#1 - 2\eta\theta\tau^2\,\#1 - \\
&\qquad 6\theta^2\tau^2\,\#1 + 6\eta\theta^2\tau^2\,\#1 + 12\eta^2\,\#1^2 - 20\eta^3\,\#1^2 - \\
&\qquad 6\eta\tau\,\#1^2 + 18\eta^2\tau\,\#1^2 - 3\theta\tau\,\#1^2 + 16\eta\theta\tau\,\#1^2 - \\
&\qquad 9\eta^2\theta\tau\,\#1^2 - 6\eta\tau^2\,\#1^2 - 6\theta\tau^2\,\#1^2 + 2\eta\theta\tau^2\,\#1^2 + \\
&\qquad 6\theta^2\tau^2\,\#1^2 - 6\eta\theta^2\tau^2\,\#1^2 + \tau^3\,\#1^2 + \theta\tau^3\,\#1^2 + \theta^2\tau^3\,\#1^2 - \\
&\qquad 4\eta^2\,\#1^3 + 20\eta^3\,\#1^3 - 16\eta^2\tau\,\#1^3 - 8\eta\theta\tau\,\#1^3 - \\
&\qquad 4\eta^2\theta\tau\,\#1^3 + 4\eta\tau^2\,\#1^3 + 4\theta\tau^2\,\#1^3 + 2\eta\theta\tau^2\,\#1^3 - \\
&\qquad 2\theta^2\tau^2\,\#1^3 + 2\eta\theta^2\tau^2\,\#1^3 - 2\theta^2\tau^3\,\#1^3 - 10\eta^3\,\#1^4 + \\
&\qquad 5\eta^2\tau\,\#1^4 + 2\eta\theta\tau\,\#1^4 + 11\eta^2\theta\tau\,\#1^4 - 2\eta\theta\tau^2\,\#1^4 - \\
&\qquad \theta\tau^3\,\#1^4 + \theta^2\tau^3\,\#1^4 + 2\eta^3\,\#1^5 - 6\eta^2\theta\tau\,\#1^5 + \eta^2\theta\tau\,\#1^6\ \&,\ 3\big]\}
\end{aligned}
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\begin{aligned}
&\{\beta \to \\
&\quad \mathrm{Root}\big[-2\eta + 4\eta^2 - 2\eta^3 + \tau - 2\eta\tau + \eta^2\tau - 3\theta\tau + 6\eta\theta\tau - \\
&\qquad 3\eta^2\theta\tau + 2\theta^2\tau^2 - 2\eta\theta^2\tau^2 + 2\eta\,\#1 - 12\eta^2\,\#1 + \\
&\qquad 10\eta^3\,\#1 + 8\eta\tau\,\#1 - 8\eta^2\tau\,\#1 + 6\theta\tau\,\#1 - 16\eta\theta\tau\,\#1 + \\
&\qquad 10\eta^2\theta\tau\,\#1 - 2\tau^2\,\#1 + 2\eta\tau^2\,\#1 + 2\theta\tau^2\,\#1 - 2\eta\theta\tau^2\,\#1 - \\
&\qquad 6\theta^2\tau^2\,\#1 + 6\eta\theta^2\tau^2\,\#1 + 12\eta^2\,\#1^2 - 20\eta^3\,\#1^2 - \\
&\qquad 6\eta\tau\,\#1^2 + 18\eta^2\tau\,\#1^2 - 3\theta\tau\,\#1^2 + 16\eta\theta\tau\,\#1^2 - \\
&\qquad 9\eta^2\theta\tau\,\#1^2 - 6\eta\tau^2\,\#1^2 - 6\theta\tau^2\,\#1^2 + 2\eta\theta\tau^2\,\#1^2 + \\
&\qquad 6\theta^2\tau^2\,\#1^2 - 6\eta\theta^2\tau^2\,\#1^2 + \tau^3\,\#1^2 + \theta\tau^3\,\#1^2 + \theta^2\tau^3\,\#1^2 - \\
&\qquad 4\eta^2\,\#1^3 + 20\eta^3\,\#1^3 - 16\eta^2\tau\,\#1^3 - 8\eta\theta\tau\,\#1^3 - \\
&\qquad 4\eta^2\theta\tau\,\#1^3 + 4\eta\tau^2\,\#1^3 + 4\theta\tau^2\,\#1^3 + 2\eta\theta\tau^2\,\#1^3 - \\
&\qquad 2\theta^2\tau^2\,\#1^3 + 2\eta\theta^2\tau^2\,\#1^3 - 2\theta^2\tau^3\,\#1^3 - 10\eta^3\,\#1^4 + \\
&\qquad 5\eta^2\tau\,\#1^4 + 2\eta\theta\tau\,\#1^4 + 11\eta^2\theta\tau\,\#1^4 - 2\eta\theta\tau^2\,\#1^4 - \\
&\qquad \theta\tau^3\,\#1^4 + \theta^2\tau^3\,\#1^4 + 2\eta^3\,\#1^5 - 6\eta^2\theta\tau\,\#1^5 + \eta^2\theta\tau\,\#1^6\ \&,\ 4\big]\}
\end{aligned}
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\begin{aligned}
&\{\beta \to \\
&\quad \mathrm{Root}[-2\eta + 4\eta^2 - 2\eta^3 + \tau - 2\eta\tau + \eta^2\tau - 3\theta\tau + 6\eta\theta\tau - \\
&\qquad 3\eta^2\theta\tau + 2\theta^2\tau^2 - 2\eta\theta^2\tau^2 + 2\eta\#1 - 12\eta^2\#1 + \\
&\qquad 10\eta^3\#1 + 8\eta\tau\#1 - 8\eta^2\tau\#1 + 6\theta\tau\#1 - 16\eta\theta\tau\#1 + \\
&\qquad 10\eta^2\theta\tau\#1 - 2\tau^2\#1 + 2\eta\tau^2\#1 + 2\theta\tau^2\#1 - 2\eta\theta\tau^2\#1 - \\
&\qquad 6\theta^2\tau^2\#1 + 6\eta\theta^2\tau^2\#1 + 12\eta^2\#1^2 - 20\eta^3\#1^2 - \\
&\qquad 6\eta\tau\#1^2 + 18\eta^2\tau\#1^2 - 3\theta\tau\#1^2 + 16\eta\theta\tau\#1^2 - \\
&\qquad 9\eta^2\theta\tau\#1^2 - 6\eta\tau^2\#1^2 - 6\theta\tau^2\#1^2 + 2\eta\theta\tau^2\#1^2 + \\
&\qquad 6\theta^2\tau^2\#1^2 - 6\eta\theta^2\tau^2\#1^2 + \tau^3\#1^2 + \theta\tau^3\#1^2 + \theta^2\tau^3\#1^2 - \\
&\qquad 4\eta^2\#1^3 + 20\eta^3\#1^3 - 16\eta^2\tau\#1^3 - 8\eta\theta\tau\#1^3 - \\
&\qquad 4\eta^2\theta\tau\#1^3 + 4\eta\tau^2\#1^3 + 4\theta\tau^2\#1^3 + 2\eta\theta\tau^2\#1^3 - \\
&\qquad 2\theta^2\tau^2\#1^3 + 2\eta\theta^2\tau^2\#1^3 - 2\theta^2\tau^3\#1^3 - 10\eta^3\#1^4 + \\
&\qquad 5\eta^2\tau\#1^4 + 2\eta\theta\tau\#1^4 + 11\eta^2\theta\tau\#1^4 - 2\eta\theta\tau^2\#1^4 - \\
&\qquad \theta\tau^3\#1^4 + \theta^2\tau^3\#1^4 + 2\eta^3\#1^5 - 6\eta^2\theta\tau\#1^5 + \eta^2\theta\tau\#1^6\ \&,\ 5]\}
\end{aligned}
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\begin{aligned}
&\{\beta \to \\
&\quad \mathrm{Root}[-2\eta + 4\eta^2 - 2\eta^3 + \tau - 2\eta\tau + \eta^2\tau - 3\theta\tau + 6\eta\theta\tau - \\
&\qquad 3\eta^2\theta\tau + 2\theta^2\tau^2 - 2\eta\theta^2\tau^2 + 2\eta\#1 - 12\eta^2\#1 + \\
&\qquad 10\eta^3\#1 + 8\eta\tau\#1 - 8\eta^2\tau\#1 + 6\theta\tau\#1 - 16\eta\theta\tau\#1 + \\
&\qquad 10\eta^2\theta\tau\#1 - 2\tau^2\#1 + 2\eta\tau^2\#1 + 2\theta\tau^2\#1 - 2\eta\theta\tau^2\#1 - \\
&\qquad 6\theta^2\tau^2\#1 + 6\eta\theta^2\tau^2\#1 + 12\eta^2\#1^2 - 20\eta^3\#1^2 - \\
&\qquad 6\eta\tau\#1^2 + 18\eta^2\tau\#1^2 - 3\theta\tau\#1^2 + 16\eta\theta\tau\#1^2 - \\
&\qquad 9\eta^2\theta\tau\#1^2 - 6\eta\tau^2\#1^2 - 6\theta\tau^2\#1^2 + 2\eta\theta\tau^2\#1^2 + \\
&\qquad 6\theta^2\tau^2\#1^2 - 6\eta\theta^2\tau^2\#1^2 + \tau^3\#1^2 + \theta\tau^3\#1^2 + \theta^2\tau^3\#1^2 - \\
&\qquad 4\eta^2\#1^3 + 20\eta^3\#1^3 - 16\eta^2\tau\#1^3 - 8\eta\theta\tau\#1^3 - \\
&\qquad 4\eta^2\theta\tau\#1^3 + 4\eta\tau^2\#1^3 + 4\theta\tau^2\#1^3 + 2\eta\theta\tau^2\#1^3 - \\
&\qquad 2\theta^2\tau^2\#1^3 + 2\eta\theta^2\tau^2\#1^3 - 2\theta^2\tau^3\#1^3 - 10\eta^3\#1^4 + \\
&\qquad 5\eta^2\tau\#1^4 + 2\eta\theta\tau\#1^4 + 11\eta^2\theta\tau\#1^4 - 2\eta\theta\tau^2\#1^4 - \\
&\qquad \theta\tau^3\#1^4 + \theta^2\tau^3\#1^4 + 2\eta^3\#1^5 - 6\eta^2\theta\tau\#1^5 + \eta^2\theta\tau\#1^6\ \&,\ 6]\}
\end{aligned}
$$