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Testing Parameter Stability in Quantile Models: An Application to the U.S. Inflation Process

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Abstract

This paper considers parameter instability tests in conditional quantile models. I suggest tests for quantile parameter instability based on the asymptotically optimal tests of Lee (2008) both in parametric and semiparametric set-up. In parametric models, Komunjer (2005)'s tick-exponential family of distributions is used as the underlying distribution, in which the test has asymptotically correct sizes even when the error distribution is misspecified. I apply our test statistic to various quantile models of the U.S. inflation process such as Phillips curve, P-star model, and autoregressive models. The test result shows an evidence of parameter instability in most quantile levels of all models. The semiparametric test rejects the stability even in more recent period with moderate economic volatility. Phillips curve model and autoregressive model have asymmetric test results across quantile levels, implying the asymmetric response of inflation to economic shocks.

Journal of Economic Literature Classification: C12, C22, E31

Keywords: Quantile Model, optimal test, parameter instability, Phillips curve, inflation

1. Introduction

The majority of economic empirical work has focused on conditional mean models. In this type of model, the relationship between X and y is described by how the mean of y changes with X solely. In general, however, covariates X may influence the conditional distribution of the response in many other ways, such as expanding its dispersion as in traditional models of heteroscedasticity, stretching one tail of the distribution, and even inducing multimodality. Explicit investigation of these effects via quantile estimation can provide a more nuanced view of the relationship, and therefore a more informative empirical analysis. In this sense, increasing attention has been devoted to quantile relationships.

In estimating and forecasting quantiles, it is crucial to investigate whether the model of interest is stable over time. Many economic factors may cause a model to become unstable. Technology shocks, changes in economic policy, and changes in economic regimes such as a shift from a closed economy to an open one are such examples. As long as the instability is not too strong, standard estimation methods are still acceptable. However, in instances of strong instability, such as the nonstationary time varying parameter and structural breaks, inference using standard methods will be misleading.

This paper introduces methods for testing stability of quantile parameters. Lee (2008)'s asymptotically point optimal test is applied to a quasi likelihood ratio (LR) based test and a semiparametric likelihood ratio test. In constructing quasi LR test, Komunjer (2005)'s tick-exponential family of distributions as the underlying distribution in which the location parameter represents the quantile level. The test is robust to more feasible circumstances, and easy to compute in that it uses a widely used quantile regression estimator under the null hypothesis while the cumbersome estimation of the density is not required to calculate the variance of the estimator. While the quasi LR test has correct asymptotic sizes, the asymptotic power would differ across quantiles when we perform the test for multiple levels of quantiles.

The semiparametric LR test treats the likelihood function as the infinite dimensional unknown nuisance parameter and uses kernel density estimates to construct the likelihood ratio function. FOr any quantile levels the semiparametric test is adaptive in the sense that the asymptotic power is equivalent to that of the LR test with correctly specified likelihood function. The adaptivity comes from the construction of the test in which the unstable parameter is asymptotically invariant to the infinite dimensional nuisance parameter that determines the shape of the underlying distribution.

The test is used to investigate the quantile parameter stability in the U.S. inflation model. Phillips curve, P-star model, and autoregressive models are considered for the testing purpose. The tests result shows an evidence of parameter instability in most quantile levels of all models. Semiparametric test rejects stability even after 1990's. In Phillips curve and autoregressive models, different quantile levels delivers different test results. Considering that the instability is mainly caused by various economic shocks, the non-identical test result implies that the inflation process has asymmetric response to the economic shocks.

2. The model and the test hypothesis

This section introduces the linear quantile model and the test hypothesis. Consider a stochastic process $(y, X) \equiv Z \equiv \{Z_t : \Omega \to \mathbb{R}^{r+1},\ r \in \mathbb{N},\ t = 1, ..., T\}$ defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ where $\mathfrak{F} = \{\mathfrak{F}_t,\ t = 1, ..., T\}$ and $\mathfrak{F}_t$ denotes the smallest σ-algebra that Z_t is adapted to, i.e. $\mathfrak{F}_t \equiv \sigma(Z_1, ..., Z_t)$. y_t is an endogenous variable with conditional distribution function $F_t(y) = Pr(Y_t \leq y_t|\mathfrak{F}_{t-1}, X_t)$ and the corresponding conditional density function $f(y_t)$, which is measurable both under the null and the alternative hypotheses. X_t is a vector of explanatory variables with the conditional density $f_X(x_t|\mathfrak{F}_{t-1})$. We are interested in the αth quantile of the distribution of Y_t conditional on the information $\mathfrak{F}_t$. Denoting this as Q_α, it is defined as

$$Q_\alpha(y_t \mid \mathfrak{F}_t) \equiv \inf_{v \in R} \{v : F_t(v) > \alpha\}$$

$$\text{or if } F_t(y) \text{ is continuous,} \quad Q_\alpha(y_t \mid \mathfrak{F}_t) \ \equiv \ F_t^{-1}(\alpha) \tag{2.1}$$

In words, conditional quantile $Q_\alpha(y_t)$ is the value that the probability of y_t being less than this value is α. In this paper I restrict my attention to a linear quantile model as

$$y_t = X_t'(\beta_0^\alpha + \beta_t^\alpha) + \epsilon_t^\alpha \quad t = 1, \ldots, T \tag{2.2}$$

where ε_t^α is the error term independent of X_t and has the following quantile restriction that $Pr(\varepsilon_t^\alpha < 0) = \alpha$. The data generating process of (2.2) is not distinctive to a common linear regression model. Indeed, (2.2) can be transformed to a traditional conditioanl mean equation as

$$y_t \ = \ X_t'\beta_t + X_t'\gamma_t\epsilon_t \tag{2.3}$$

where β_t is the mean parameter and $X_t'\gamma_t$ represents the heteroscedasticity of the error term. It subsumes many models of systematic heteroscedasticity which have appeared in the econometrics literature: Goldfeld and Quandt (1965)'s model($\sigma(x) = \sigma x_k$) is a special case when $\gamma_t = \sigma e$, and Harvey (1976) and Godfrey (1978)'s multiplicative heteroscedasticity model can be considered as a case when $\sigma(x) = x\gamma + o(||\gamma||)$. The conditional quantile of y_t in the model (2.3) is then simply $Q_\alpha(y_t \mid \mathfrak{F}_t) \equiv X_t'\beta_t + X_t'\gamma_t \cdot Q_\alpha(\varepsilon_t)$. Consequently, the αth conditional quantile of y_t can be expressed as a linear function of X_t,

$$Q_\alpha(y_t \mid \mathfrak{F}_t) \equiv X_t'[\beta_t + \gamma_t Q_\alpha(\varepsilon_t)] = X_t'\beta_{\alpha,t} \tag{2.4}$$

where $\beta_{\alpha,t} = \beta_t + \gamma_t Q_\alpha(\epsilon_t)$. Hence, the quantile parameter β_α is determined by the mean parameters β, the scale parameters γ and other parameters that determine the shape of the conditional distribution Q_α.

This comparison provides another motivation to pay much attention to the quantile estimation method is that the conditional mean model is insufficient to make inferences about the risks of the variable of interest. The measurement and management of risk has been an important issue in finance and macroeconomics. There is no doubt that the quantification of the tradeoff between risk and expected return is one of the main problems in finance, which makes the estimation of risk of central importance. In addition, in many of the central banks, density forecasts of inflation are preferred to point forecasts in the sense that the former contains the uncertainty structure of the forecast. Conditional quantiles have more information than just the conditional mean in that they contain other information about the uncertainty structure of the variable of interest such as skewness, kurtosis and any other factors that determine the shape of the distribution. Consequently, quantile estimation provides a better way to measure risks.

The objective of this paper is to test whether the unstable parameter process $\{\beta_t^\alpha\}$ presents in the model. Under the null hypothesis of stability, $\{\beta_t^\alpha\}$ are zeros for all $t = 1, \ldots, T$, such that the parameter vector would be β_0^α. Under the alternative $\{\beta_t^\alpha\}$ follows some permanent breaking processes.[1] The alternative hypothesis is not defined in a single form because there exist a large variety of ways in which θ_t is not stable. Any specific assumption on unstable β_t^α would lead to a different alternative hypothesis resulting in a different testing problem. Existing instability tests can be categorized into two big streams based on types of unstable processes: One is the test of structural breaks, and the other is the test of time varying parameters. Structural break tests consider a model in which θ_t permanently shift N times in a sample period. For a single break example, the parameter vector equals β_0^α) for $t = 1, \ldots, \tau$ and $\beta_0^\alpha + \bar{\beta}^\alpha$ for $t = \tau + 1, \ldots, T$. Time varying parameter tests posit random process of β_t^α. Even within time-varying parameter approaches there are many possible alternatives based on the distributional properties of β_t^α. However, economic theory generally does not provide enough information to pick a specific alternative process. Consequently, an alternative process are often arbitrarily chosen depending on what the researcher has in mind.

Elliott and Müller (2006), Nyblom (1989), and Lee (2008) get around the problem by providing only minimal identifying conditions on the unstable process. Their idea is that the seemingly different approaches of structural breaks and time varying parameters are in fact not distinctive, but are considered as specific forms of a unified framework. For example, if we let $\Delta\beta_t^\alpha$ have a continuous distribution with probability p and equal zero with probability $(1 - p)$, then this time varying parameter process is reduced to a multiple structural break with $(T \cdot p)$ expected breaks. On the other

[1]Note that this paper consider unstable parameter processes as the only alternatives, which excludes transitory change in parameters, stable random coefficients, etc, in that the traditional estimation methods are still valid under stable time varying parameter processes.

hand, it becomes a random walk if $\Delta\beta_t^\alpha$ is *iid* normal. This paper follows their idea and define the local alternative process $\{\beta_t^\alpha\}$ as

Condition 1. *The unstable process $\{\beta_t^\alpha\}$ satisfies that*

$$T\Omega^{1/2}\beta_t^\alpha \Rightarrow W(r)$$

where $r = t/T$, $W(r)$ is the $k \times 1$ multivaraite standard Wiener process, and Ω is the gloval covariance matrix of β_t^α

Condition 1 is shown to be weak enough to cover a variety of permanent breaking processes such as many or relatively few breaks, clustered breaks, regularly occurring breaks, or smooth transitions to changes in the parameters. (See Elliott and Müller (2006) and Lee (2008) for details.) However, Nyblom (1989)'s test is locally most powerful only under the counterfactual assumption that θ_0 is known, and Elliott and Müller (2006)'s optimality is restricted only to linear regression models with Gaussian error distribution. Lee (2008) provide an asymptotically point optimal tests for a variety of general nonlinear models. Consequently, I use Lee (2008) in this paper and show that the test can be applied to quantile linear regression model in the next section.

Another issue of quantile parameter break tests arises when we perform the tests for multiple quantile levels. If a quantile forecast is used for interval forecast or density forecast such as inflation fan chart it is interesting to examine if there is an asymmetry in the processes at upper and lower quantile levels. For example, one can suspects that Fed's monetary policy would have asymmetric effect on the inflationary risk and the deflationary risk, or the policy itself would be asymmetric due to Fed's asymmetric loss function. This asymmetry would consequently stretches one tails of the distribution while keeping the other tails stable. In this case, we would expect that breaks would not occur identically across all quantile levels. As is shown in (2.4), using distribution function with finite-dimensional parameters such as **?**, and Nyblom (1989) would not capture the heterogeneous pattern of the unstable process, and the underfitting problem of parametric models would accordingly come to the fore. Instead, semiparametric approach in which we allow for infinite dimensional parameters provides a way to capture the complexity of multiple quantile level tests. Moreover, Lee (2008) suggests a way that the parameter instability tests in semiparametric setup is adaptive, i.e. its asymptotic power function is equivalent to the asymptotic power envelope when the underlying distribution is correctly specified. It make us predict that each parameter instability test of multiple quantile levels would achieve the best asymptotic power. Next section supports the conjecture under suitable conditions.

In order to construct the test function. We impose on the distributional property in the model as follows.

Condition 2. *i) ϵ_t is iid and conditionally independent of X_t given $\mathfrak{F}_{t-1}$. The error distribution $\{g(\epsilon_t)\}$ does not depend on θ_t in the null hypothesis.*

ii)

beta$_0^\alpha$ is an interior point of the parameter space Γ.

iii) X_t has conditional distribution $f_X(X_t|\Im_{t-1})$ with respect to some σ-finite measures, $\{f_X(X_t|\Im_{t-1})\}$ does not depend on parameters β_0^α and β_t^α for all $t = 1, \ldots, T$.

iv) Under H_0, $\{X_t\}$ are mixing with either ϕ of size -r/2(r-1), r $\geqq$2 or α of size -r/(r-2), r> 2.

v) Under H_0, $E\left[|X_{t,i}|^r\right] < \Delta < \infty$ for all $t = 1, \ldots, T$ and $i = 1, \ldots, k$.
$T^{-1}\sum_{t=1}^{[sT]} X_t X_t' \to sJ_X$ uniformly in s where $J_X = E[X_t X_t']$. $T^{-1}\sum_{t=1}^{T} X_t X_t'$ is uniformly positive definite.

3. Testing Parameter Stability in Quantile Models

This section describes methods to perform tests for quantile parameter instability. Let $\dot{\ell}_i^* = (\dot{\ell}'_{i,1}, \ldots, \dot{\ell}'_{i,T})'$ be the first derivative of the log likelihood function with respect to i^{th} element of β_t^α, which is normalized to have unit variance. Let's define ζ_i as the $T \times 1$ vector of the partial sum of $\dot{\hat{\ell}}_{i,t}^* = \dot{\ell}_{i,t}^* - \frac{1}{T}\sum_{\tau=1}^{T} \dot{\ell}_{i,\tau}^*$, i.e. j^{th} element of ζ_i is $\sum_{t=1}^{j} \dot{\hat{\ell}}_{i,t}^*$. This paper uses Lee (2008)'s test function which is defined as

$$B(\Omega) = \sum_{i=1}^{k} \zeta_i' \left[\frac{a_i^2}{T^2} I_T - F M_e F' \right]^{-1} \zeta_i \tag{3.1}$$

where $F = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & \ldots & 1 \end{pmatrix}$, $M_e = I_T - \frac{1}{T}ee'$, e is the $T \times 1$ vector of ones, and a_i's represent the eigenvalues of Ω after adjusted by the size of $\{X_t\}$. The test function is derived to be asymptotically equivalent to the increasing transformation of the likelihood ratio which is generally defined as $\int \frac{f_1(\epsilon_t)}{f_0(\epsilon_t)} d_{\nu_\beta}$, where $f_1(\cdot)$ and $f_0(\cdot)$ are the density function under H_1 and H_0, respectively, and ν_β is the probability measure of $\{\beta_t^\alpha\}$. The measure ν_β characterizes the shape of the unstable β_t^α, and the tests generally depends on ν_β. However, Lee (2008) shows that the likelihood function with any specific unstable $\{\beta_t^\alpha\}$ measures $\nu)_\beta$ are asymptotically equivalent to a certain increasing function of $B(\Omega)$ as long as $\{\beta_t^\alpha\}$ follows Condition 1. Consequently, $B(\Omega)$ is asymptotically most powerful if the likelihood function is correctly specified by Newman-Pearson lemma, and we can use $B(\Omega)$ without worrying about misidentifying the alternative process.

In order to construct the test $B(\Omega)$, it is required to estimate $\{\dot{\ell}_t^*\}$ and thereby $\{\zeta_t\}$. Depending on the way that $\{\zeta_t\}$ are calculated, I suggest two different types of tests using $B(\Omega)$; quasi LR (QLR) test, and semiparametric LR test. QLR test uses specific likelihood function to estimate ζ_t so that the test has the same asymptotic null distribution as the genuine LR test in which the error distribution is correctly identified. The QLR test, however, is not generally asymptotically optimal because the asymptotic equivalency does not hold under the alternative hypothesis unless the employed distribution is equivalent to the true one . Semiparametric LR test estimates ξ_t using nonparametric methods such as kernel estimates of the density function. Lee (2008) shows that under suitable conditions, the semiparametric test are asymptotically equivalent to the genuine parametric LR test both under the null and the alternative hypothesis, which implies that the test is asymptotically most powerful.

This paper uses Komunjer (2005)'s tick-exponential family of distributions to construct a QLR test, which is defined as

$$\varphi_t^\alpha(y_t|X_t,\beta^\alpha) =$$

$$\exp\left[-(1-\alpha)\{a_t(X_t\beta^\alpha) - b_t(y)\}1_{\{y<X_t\beta^\alpha\}} + \alpha\{a_t(X_t\beta^\alpha) - c_t(y)\}1_{\{y\geq X_t\beta^\alpha\}}\right] \tag{3.2}$$

where $a_t(\cdot)$ is continuously differentiable and $a_t(\cdot), b_t(\cdot), c_t(\cdot)$ are functions such that (i) φ_t^α is a probability density, i.e. $\int_{\mathbb{R}} \varphi_t^\alpha dy = 1$; (ii) $X_t\beta^\alpha$ is the α-conditional quantile of φ_t^α. For a given value of probability α $\varphi_t^\alpha(\cdot)$ is interpreted as exponential by parts where the two parts have different slopes, proportional to $1-\alpha$ and α, respectively. The special case is an asymmetric type of a *Laplace distribution* by defining $a_t(X_t\beta) =$

$[1/(\alpha(1-\alpha))]X_t\beta$, and $b_t(y) = c_t(y) = [1/(\alpha(1-\alpha))]y$, i.e.,

$$\varphi_t^{\alpha,L}(\cdot) = \exp[\frac{1}{\alpha}(y_t - X_t\beta)1_{\{y_t<X_t\beta\}} - \frac{1}{1-\alpha}(y_t - X_t\beta)1_{\{y_t\geq X_t\beta\}}] \tag{3.3}$$

The distribution (3.3) is proportional to an asymmetric slope function and is reduced to the Laplace distribution when α is 0.5. This type of likelihood function, as proposed by Komunjer (2005), is advantageous for analyzing conditional quantile models. First, parameters in (3.2) represent conditional quantile parameters. In distributions with parameters as functions of mean, variance, and other moments (such as the normal and t-distribution), A re-parameterization is required in order to present them as a function of quantile parameters. As shown before, this re-parametrization makes the quantile parameter a function of the shape of the distribution. This hinders the use of the condition that the distribution is stable across breaks. However, since the parameters of $\varphi^\alpha(\cdot)$ represent the conditional quantile, the shape of the distribution may stay stable over different variations of parameter β. As a result, we can easily construct the likelihood ratio test statistic.

Another merit of this function is that maximum likelihood estimators based on this likelihood are consistent even if the likelihood function is misspecified. Komunjer (2005) shows that the MLE is consistent if any form of the tick-exponential type likelihood function is used. Koenker and Bassett (1978)'s quantile regression method is regarded as MLE when the likelihood function is (3.3). This property implies that, as pointed out by White (1982), even though the test statistic is not optimal due to the misspecification, the test statistic is still reasonable if we correct some problems. Proposition 1 shows that the quantile parameter instability test $B(\Omega)$ based on $\varphi^\alpha(\cdot)$, denoted as $B^Q(\Omega)$ has correct asymptotic null distribution which makes $B^Q(\Omega)$ QLR. (3.2) is not differentiable at points where either $y_t = X_t'\bar{\beta}$ or $y_t = X_t'\bar{\beta} + X_t'\beta_t^\alpha$, which hinders the use of the first derivative to calculate ζ_i. However, it is easy to show that (3.2) is quadratic mean differentiable so that a similar approximation as Taylor expansion is possible in this case. Then $\{\dot{\ell}_t^*\}$ can be defined as the function of the *Hellinger derivative*, i.e. $\{\dot{\ell}_t^*\} = J^{-1/2}\{\dot{\ell}_t\}$, and

$$\dot{\ell}_t = \frac{\dot{a}_t(\cdot)}{1-\alpha} X_t - \frac{\dot{a}_t(\cdot)}{\alpha(1-\alpha)} X_t 1_{\{y_t < X_t \beta_\alpha\}}$$

$$J = \dot{a}_t(\cdot)^2 \cdot \left[\frac{1}{(1-\alpha)^2} + \frac{1-2\alpha}{\alpha(1-\alpha)^2}\right] E[X_t X_t'] \tag{3.4}$$

where $\dot{a}_t(\cdot)$ is the first derivative of $a_t(\cdot)$. In calculating the feasible test function, β^α should be replaced by the maximum likelihood estimator. Komunjer (2005) suggests the method of calculating MLE using 'minimax' approach. However, if we derive a test using (3.3), MLE is equivalent to the widely known Koenker and Bassett (1978)'s Quantile Regression Estimator and it make us easy to compute the test statistic $B(\hat{\Omega})$. [2]

The maximum likelihood estimation of β_α using (3.3) solves a linear programming problem such as "$\min_z c'z$ subject to $Az = y,\ z \geq 0$, where $A = (X, -X, I_T, -I_T)$, $y = (y_1, ..., y_T)'$, $z = (\beta^{+\prime}, \beta^{-\prime}, u^{+\prime}, u^{-\prime})'$, $c = (0', 0', \theta \cdot e', (1-\theta) \cdot e')'$, $X = (X_1, ..., X_T)$,, and subscript + and - are defined as for any a, $a^+ = max[a, 0]$ and $a^- = -min[a, 0]$.

Buchinsky (1997a) and Koenker and Park (1996) suggest its solution methods and show the uniqueness of the solution. traditionally there are two methods to solve the linear programming problem. One is to travel from vertex to vertex along the edges of the polyhedral constraint set, choosing at each vertex the path of steepest

[2] The problem of choosing the density function that delivers the best power is not addressed here. When the distribution function is correctly specified, the error density with maximal fisher information function provides the best asymptotic power. But it does not necessarily mean that the test would provide the best power under misspecified circumstance. In this sense we choose the asymmetric Laplace density which gives the easiest way to construct a test statsitic.

descent, until we arrive at the optimum. The other is to take Newton steps from the interior of a deformed version of the constraint set toward boundary. Recently, Buchinsky (1997b) suggests GMM estimation by letting the first order conditions of the minimization problem be the moment conditions.

Let $\hat{\beta}^{\alpha}$ be the MLE of β^{α} and $\hat{B}(\Omega)$ be the plug-in version of $B(\Omega)$ where β^{α} is replaced by $\hat{\beta}^{\alpha}$. $\hat{\beta}^{\alpha}$ is estimated either by the method of Komunjer (2005) or the quantile regression coefficient in case where the likelihood function is (3.3). The following proposition provides the asymptotic properties of $B^{Q}(\Omega)$

Proposition 1. *Suppose the quantile model* (2.2) *satisfies Condition 2 , and let* $\hat{\psi}_T(Z|\Omega)$ *be a critical function for* $\hat{B}^{Q}(\Omega)$.

(1) $\lim_{T\to\infty}\int \hat{\psi}_T(Z|\Omega) f_0(Z|\beta_0^{\alpha})dZ = \alpha$

(2) Let $\hat{\phi}_T(Z|\Omega)$ *be the test function of any feasible parameter instability tests for Condition 1 process that have limiting size* α *for all values at* $T-$*neighborhood of zero vector. If the underlying distribution is correctly specified in* $\hat{B}(\Omega)$,

$$\overline{\lim_{T\to\infty}}\int \hat{\phi}_T(Z|\Omega) f_1(Z|\beta_0^{\alpha},\beta^{\alpha})dZ \leq \underline{\lim_{T\to\infty}}\, t \int \hat{\psi}_T(Z|\Omega) f_1(Z|\beta_0^{\alpha},\beta^{\alpha}) \equiv \Psi(\Omega)dZ$$

The first part of the proposition indicates the $B^{Q}(\Omega)$ is asymptotically valid regardless of the true error distribution. The second part indicates that the feasible test $B^{Q}(\Omega)$ with β_0^{α} has the best asymptotic power among all feasible test functions under unknown β_0^{α}. It implies that the test is derived based on the least favorable hypothesis in which the local direction of β_0^{α} is set to have the minimal asymptotic power envelope. Note that the test has optimal power no matter what the specific form of the unstable process is. That is, knowledge of the unstable process is asymptotically irrelevant as long as the unstable process satisfies some weak condition. The optimality of $\hat{B}^Q(\Omega)$ can be obtained under the counterfactual circumstance that the error term is unknown. Also when we perform the multiple levels of quantiles, it is impossible to pertain asymptotic optimality for every quantile levels. In this sense, it is better in practice to regard $B^Q(\Omega)$ as robust tests for multiple quantile levels.

On the other hand, it is shown to be amenable to obtain the asymptotic optimality for every levels of quantiles when we consider semiparametric circumstances, in which the underlying distribution is admitted to be unknown but is treated as an unknown infinite dimensional nuisance parameter. Lee (2008) derives the semiparametric power envelope which is sharp in a general setup and shows that it is asymptotically equivalent to the parametric power envelope. That is, the semiparametric test would achieve the same asymptotic optimal power as if we know the true error distribution. The idea is as follows: Consider a parametric submodel in which the error distribution is known to belong to a specific parametric family of distribution indexed by a finite dimensional unknown nuisance parameter η. If the density function is differentiable in quadratic mean, the likelihood function can be asymptotically equivalent to

$$\widetilde{LR}_T = \int exp\left[\sum \dot{\ell}_t' \beta_t^\alpha - \frac{1}{2}\beta_t^{\alpha'} J \beta_0^\alpha - \frac{1}{2}\beta_0^{\alpha'} J_\eta \eta + \sum \dot{\ell}_t^{\eta'} \eta - \frac{1}{2}\eta_0^{\alpha'} J_{\eta\eta}\eta\right] d_{\nu_\beta} \tag{3.5}$$

where $\dot{\ell}^\eta$ accounts for the first derivative with respect to η, $J_\eta = E[\dot{\ell}_t \dot{\ell}_t^\eta]$, and $J_{\eta\eta} = E[\dot{\ell}_t^\eta \dot{\ell}_t^{\eta'}]$. If the cross term $\frac{1}{2}\beta_0^{\alpha'} J \beta_0^\alpha$, then we can isolate the terms related to η out of (3.5) which implies that the construction of the LR test with eliminating the effect of η is possible. Stein's necessary condition deals with this problem by providing condition that $J_\eta = 0$. But Lee (2008) deals with the problem by showing that the test is invariant to the reparameterization of β_t to $\beta_t - \sum_{i=1}^T \beta_i$ which implies that the first derivative $\dot{\ell}_t$ can be replaced by $\dot{\ell}_t - \sum_{i=1}^T \dot{\ell}_i$. It makes $\frac{1}{2}\beta_0^{\alpha'} J \beta_0^\alpha$ be zero so that the cross term is successfully eliminated from (3.5), and the test in the parametric submodel has the same asymptotic power envelope in parametric models. Since this argument holds for all kind of submodels, the semiparametric power envelope which is defined as the infimum of those of parametric submodels is also equivalent to parametric one.

The feasible test that reach the power envelop is possible if there exist a $\sqrt{T}$-consistent regular estimator of β_0^α and the traditional quantile regression estimator is shown to be a candidate. A possible construction of the efficient estimator of $\{\dot{\ell}_t\}$ is to use a kernel estimation method. Using data and the consistent estimator of θ_0, compute the residuals $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_T$ with $\tilde{\epsilon}_t = \epsilon(y_1, \ldots, y_t, X_1, \ldots, X_T, \widehat{\theta})$ for $t = 1, \ldots, T$. A kernel density estimator is defined as for all e in a small neighborhood of each value of $\tilde{\epsilon}_t$

$$\hat{f}_T^\epsilon(e; \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_T) = \frac{1}{(T-1)a_T} \sum_{i \neq t} k\left(\frac{e - \tilde{\epsilon}_i}{a_T}\right) \tag{3.6}$$

$$\hat{f}_T^{\epsilon\prime}(e; \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_T) = \frac{1}{(T-1)a_T^2} \sum_{i \neq t} k'\left(\frac{e - \tilde{\epsilon}_i}{a_T}\right) \tag{3.7}$$

where a_T is a bandwidth and the kernel $k(\cdot)$ is three times continuously differentiable with derivative $k^{(i)}$ satisfying $\|k^{(i)}(z)\| < ck(z)$ with $i = 1, 2, 3$ for some positive c, and $\int z^2 k(z) dz < \infty$. $\{\hat{\dot{\ell}}_t^\theta\}$ and $\hat{J}_\theta$ are defined as

$$\hat{\dot{\ell}}_t(\tilde{\epsilon}_t; \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_T) = X_t \hat{\dot{\ell}}_t^\epsilon(\tilde{\epsilon}_t; \cdot) \frac{\hat{f}_T'(\tilde{\epsilon}_t; \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_T)}{b_T + \hat{f}_T(\tilde{\epsilon}_t; \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_T)} \tag{3.8}$$

$$\hat{J} = \frac{1}{T} \sum_{t=1}^T \hat{\dot{\ell}}_t(\tilde{\epsilon}_t) \hat{\dot{\ell}}_t(\tilde{\epsilon}_t)' \tag{3.9}$$

where $\{b_T\}$ is a sequence of constants such that $(T a_T^3 b_T)^{-1} \to 0$. Let $B^*(\Omega)$ be the test function using (3.8) and (3.9), and $\psi_T(Z|\Omega) = 1_{[B^*(\Omega) > k_\alpha]}$ be its critical function, where k_α is the continuous function satisfying $E_0[\psi_T(Z|\Omega)] = \alpha$. Let

$\Psi^*(\omega) = \underline{\lim}_{T\to\infty} \int\int \psi_T(Z|\Omega) f_1(Z|\eta) dZ d\nu_\delta$. The following proposition shows that $B^*(\Omega)$ is adaptive.

Proposition 2. *Suppose the error distribution $g(\epsilon)$ is differentiable in quadratic mean. Then under condition 1), any asymptotically similar test $\phi(Z|\Omega)^*_T$ satisfies*

$$\overline{\lim_{T\to\infty}} \int \phi^*_T(Z|\Omega) f_1(Z|\eta) dZ \leq \Psi^*(\Omega) = \Psi(\Omega)$$

Consequently, $B^*(\Omega)$ based on the kernel density estimation provides the best asymptotic power for every parameters across all quantile levels. Note that $B^Q(\Omega)$ and $B^*(\Omega)$ is derived based on the assumption that Ω is known, which is unobservable in practice. Consequently, there is no uniformly most powerful test in this framework. Instead, if we focus on one point in the alternative parameter space, we can find a most powerful test in the neighborhood of the predetermined point. Such a test is called a *point optimal test.* In this test we follow Elliott and Müller (2006) and Lee (2008)'s suggestion that $\hat{\Omega} = c^2 I_k$ where $c = 10.$ as Elliott and Müller (2006). In constructing both $B^Q(\Omega)$ and $B^*(\Omega)$, The proposed algorithm to construct an asymptotically efficient test statistic for both $B^Q(\hat{\Omega})$ and $B^*(\hat{\Omega})$is as follows.

Step 1) Estimate the initial parameter β_0^α under H_0 denoted as $\hat{\beta}_0^\alpha$. Any method of the estimation such as QMLE, and quantile regression is possible as long as $\hat{\beta}_0^\alpha$ is consistent. Calculate the residuals $\tilde{\epsilon}_t = y_t - X_t'\hat{\beta}_0^\alpha$

Step 2) Estimate $\hat{\dot{\ell}}_t$ and $\hat{J}_T$ and thereby $\hat{\dot{\ell}}^*_t = \hat{J}_T^{-\frac{1}{2}} \hat{\dot{\ell}}_t$ by either using either (3.8), (3.9) (for $B^*(\cdot)$ or (3.4) (for $B^Q(\cdot)$. Denote i^{th} elements of $\{\hat{\dot{\ell}}^*_t\}$ by $\{\hat{\dot{\ell}}^*_{t,i}\}$, $i = 1, \ldots, k$.

Step 3) For each $\{\hat{\dot{\ell}}^*_{t,i}\}$, generate a new variable, $\hat{w}_{t,i} = r\hat{w}^*_{t-1,i} + \Delta\hat{\dot{\ell}}^*_{t,i}$ and $\hat{w}_{t,1} = \hat{\dot{\ell}}^*_{t,1}$.

Step 4) Regress $\{\hat{w}_{t,i}\}$ on $\{r^t_{a_i}\}$ for each i to get the sum of squared residuals where $r_a = 1 - aT^{-1}$. Sum all of those over $i = 1, \ldots, k$.

Step 5) Multiply this sum by r, and subtract it from $\sum_{i=1}^k \sum_{t=1}^T \hat{\hat{\ell}}^*_{t,i}$.

4. Quantile Models in Inflation Process

Quantile models of inflation is arousing more attention for both economics and policy making. It is importantly used as an alternative method of density forecasts which are being increasingly used in practice. (See Tay and Wallis (2000) for a survey of application in macroeconomics and finance.) Point forecasts, namely the central tendency of the forecasts, are currently the most widely used methods, but are being increasingly criticized in that they contain no description of the associated uncertainty.

Density forecasts of inflation is estimates of the probability distribution of its possible future values. They provide a description of forecast uncertainty, and act as supplement to the point forecast in that the point forecast is considered as the central points of ranges of forecast uncertainty. Since density forecasts estimate a complete description of the uncertainty structure, they can be seen to provide information on all possible intervals and quantiles. However, density forecasts generally require the function form of the density to be specified or complicate nonparametric estimation of the density which sometimes has poor forecasting power in relatively small samples. Quantile forecast can avoid fully nonparametric method and has the advantage of not requiring the density and will thus be more robust to certain types of misspecification such as tail behavior of the distribution. Sometimes it is enough to obtain a finite levels of quantile for certain forecasting purposes. In addition, even though the purpose is the density forecast, Thompson and Miller (1986) show that a natural way to summarize the predictive distribution is by presenting selected quantiles.

An example of quantile type forecasts of inflation is the U.S. Survey of Professional Forecasts (SPF), known as the ASA-NBER survey. In this survey, forecasters are asked not only to report the point forecast and forecast horizons, but also to attach the density forecasts for inflation and output growth. In each case, a number of bins, in which the future value of output/inflation might fall are pre-assigned, and each survey respondents is asked to report their associated forecast probabilities. The forecast is thus represented as a histogram on a preassigned grid which is associated with the inverse of forecast quantiles.

The second example is the Bank of England Monetary Policy Committee's density forecast of inflation, known as the inflation fan chart. The density forecast is represented graphically as a set of prediction intervals, covering 10, 20, ...,90 percentiles of probability distribution. The lighter shades are for the bands further from the mode (or the median). Since the distribution becomes increasingly dispersed, the quantiles fan out as the forecast horizon increases. The original Bank of England's fan chart chooses the mode of the density forecast as its preferred central projection. Thus the central tendency is apart from the median when the predicted distribution is asymmetric. It implies that the graphed prediction intervals do not coincide with quantiles so that, for example, 90 percent prediction intervals is not formed by 5 percent and 95 percent quantiles under asymmetry of distribution. Wallis (1999) gets around the problem by suggesting an alternative fan chart based on central prediction intervals.

The fan chart is analytically drawn by choosing a particular probability density function. Once the values have been assigned to the underlying parameters in the density function, probabilities can be readily calculated. The density function Bank of England uses is the skewed version of the normal distribution, called two piece normal, in which an additional parameter describing the asymmetry of the distribution is introduced. However, it is well known that tails of normal distribution is too thin to adequately describe inflation and other macroeconomic data. In addition, the two piece normal density has only 3 parameters to describe the whole distribution. Consequently, each eighteen quantile levels are determined by 3 parameters.

The problem can be overcome either by using nonparametric forecasting of density (Fujiwara and Koga (2002)) or by directly forecasting quantile. Taylor and Bunn (1999) apply quantile regression approach to generating forecast intervals of various macroeconomic data to show that the quantile forecasting method is encouraging.

Bank of England's fan chart is the combination of model based point forecast and the Monetary Policy Committee (MPC)'s subjective probability forecast. Once the point forecast and the variance are estimated based on the macroeconomic model, MPC judges whether the uncertainty will be increased and whether the upward and downward risks are balanced. Thus the fan chart explicitly allowed the structural breaks at the time of the forecast. However, the model is estimated based on the assumption that the parameters are constant, leading to a conflict that an acknowledged instability at time t is reversed to be stable at any time periods later than t. Cogley, Morozov, and Sargent (2005) get around this problem by using a Bayesian vector autoregression in which the parameters follow driftless random walks. Vega (2005) also Bayesian method where the parameters are allowed to be non constant but assumes they are stable.

5. Testing Quantile Parameter Stability of

U.S. Inflation Process

This section explores the empirical evidence of the instability in quantile models for the U.S. inflation process. Structural break is a widely accepted phenomena in models for inflation, and many researchers have devoted much effort to identifying structural breaks in the inflation processes. Clak and McCracken (2006) find a break in 1982, while Estrella and Fuhrer (2003) suggest another break in 1984. Jouini and Boutahar (2003) find evidence of a structural break in the AR coefficient in 1990. These researches find no break after the 1990's. On the other hand, recent studies such as Stock and Watson (2006), and Atkeson and Ohanian (2001) argue that economic relationships have become unstable, even in recent years, so that the predictive powers of the models for forecasting inflation are still doubtful, even after the reduction of the volatility in the great moderation. Those tests are performed for the mean parameter stability and occasionally the variance stability. No works have been devoted to the instability of inflation quantiles.

The test in this section serves three purposes. First, I analyze whether assuming unstable parameters is justified by the data. The test will provide insights on how to choose a relevant model. Second, I analyze if the commonly used sample splitting method overcomes the instability problem in the presence of structural break. For example, it is generally perceived that around 1982 there were significant changes in macroeconomic behaviors, such as the investment prices' average rate of decline, the conduct of monetary policy, macroeconomic volatility and the regulatory environment. Most studies explicitly considering the break generally make effort to overcome the problem by reestimating the model on split subsamples. Gali, López-Salido, and Vallés (2003), Fisher (2006), Clarida, Gali, and Gertler (2000), and Barth and Ramey (2005) suggest to split into pre-Volcker period (post war-1979) and Modern era (1982 to the present). With this insight, I examine whether the split sample method could resolve the instability problem by performing the stability tests to each split sample period. The more recent subsample period (1991-current) is also considered to evaluate the recent debates about whether the inflation forecast models become more stable after 1990s.

Third, I observe whether the breaking processes are identical for all levels of quantile. This will provide an inference on the asymmetry of policy effect in that non-identical test results implies the different responses of the inflation to policy variables associated with different points on its conditional distribution. It will also give an implication of the quantile forecasting method. Most of the literatures on inflation quantile forecasts, especially the inflation fan chart, are based on specific parametric distributions in which a few number of parameters determine the shape of the distribution. In this case, for any level of quantile, quantile parameters are functions of these few parameters only. Thus a breaking process in one parameter would affect all levels of quantiles. This implies that the test should have identical results at all quantile levels. Consequently, if a model has different test results across different levels of quantiles, it would provide an evidence of inadequacy of the parametric distributional assumption. In addition, different test results will give an inference about potential asymmetry of monetary policy and business cycle. Analysis of different levels of quantiles provides a broader picture of inflation process against a various external and policy shocks. To perform the test, I consider various types of predictive models which are listed below.

Phillips Curve(BPC)	$: \pi_t = \beta_0 + \beta_1(L)\pi_t + \beta_2(L)\hat{g}_t + \epsilon_t$
P-star Model	$: \pi_t = \eta^\alpha \ln\left(\frac{P_{t-1}}{P^*_{t-1}}\right) + \beta_1(L)\pi_t + \epsilon^\alpha_t$
	$: P^*_t = \frac{M_t V^*_t}{y^*_t}$
Autoregressive Model(AR)	$: \pi_t = \beta_0 + \beta_1(L)\pi_t + \epsilon_t$

π_t = inflation rate	$\hat{g}_t$ = NAIRU gap
m_t = money supply (M2)	y_t = output (industrial production)

where $X(L)$ represents the lag operator, and P^*, V^* and y^* indicate their longrun equilibrium level, respectively. BPC is a backward-looking statistical Phillips Curve which is believed by many to be the preferred tool for forecasting inflation. (See Stock and Watson (1999).) On the other hand, recent research such as Stock and Watson (2006), and Bachmeier and Swanson (2005) cast doubt on the predictive power of the Phillips Curve model. P-star model has first proposed by Federal Reserve Board as a simple inflation forecasting method. It uses money equation as a longrun relationship of inflation, output, and money in which P-star is interpreted as the longrun price level. However, the forecasting power of P-star model is cast doubt on by many researchers, especially suspected on the stable relationship between inflation and money quantity. (See Bachmeier and Swanson (2005), and Christiano (1989).) The AR model has an advantage and has been used by many in that it is simple and has a good forecasting power.

The data spans the period between January 1962 and October 2007. I also consider a couple of subsamples; Jan.1982-Oct.2007, and Jan.1991 to Oct.2007. The split period is chosen based on the findings of a structural breaks by the current researches. Tests in the later period will provide implications on the stability after the great moderation.

I use urban CPI to calculate inflation, π_t, and industrial production index for the output, y_t. M2 is used for the money supply. Unemployment is the civilian unemployment rate. NAIRU, y_t^* and V_t^* are estimated by Hodrick-Prescott filtering. The lags are determined by Akaike Information Criteria(AIC) and all data are seasonally adjusted.

Table 5 shows the test results. The adaptive test $B^*(\Omega)$ shows strong evidence of instability of Phillips Curve in all sample periods, while the test $B(\Omega)$ could not reject the instability in the upper quantile parameters of the sample periods 1962-2007, 1982-2007, and all quantile levels in the recent period (1991-2007). Throughout all the models, the parametric test B has a tendency not to reject the stability compared to $B^*(\Omega)$. Note that $B(\Omega)$ does not have asymptotic optimality because different quantile levels use different likelihood function although they are all from the same data generating process. Considering that $B^*(\Omega)$ is asymptotically optimal, we would expect that the different test results are from the power property of the tests rather than the small sample size distortion, which implies that $B(\Omega)$ result is more favored. This inference becomes clearer if we compare the them with the test of the mean and variance stability. Table 2 shows the $B^*(\Omega)$ test result for the mean and variance parameters using the same model with the same data. The test could not reject the mean stability in the sample period 1991-2007, which is similar to $B(\Omega)$ test result for quantile instability. But the variance presents instability in all subsample period. As noted in section 2, quantile parameter can be interpreted as the function of not only the mean but also the variance and other distributional behaviors, which implies that any breaks in mean or variance cause breaks in quantiles. In this regards $B^*(\Omega)$ test is more associated with the mean-variance tests.

Table 1. Test Results: Stability of the U.S. Inflation Process

α (quantile)	B			B		
	1962-2007	1982-2007	1991-2007	1962-2007	1982-2007	1991-2007
	Philips Curve					
0.30	$39.11^{\dagger}$	$51.10^{\dagger}$	17.20	$57.40^{\dagger}$	$53.54^{\dagger}$	21.43
0.40	32.13^{**}	$36.22^{\dagger}$	23.85	$50.26^{\dagger}$	$46.72^{\dagger}$	28.62^{**}
0.50	28.97^{*}	28.84^{**}	23.99	$52.95^{\dagger}$	$51.67^{\dagger}$	26.82
0.60	26.77	27.66	18.64	$50.15^{\dagger}$	$55.35^{\dagger}$	32.68^{**}
0.70	21.33	23.45	12.71	$60.94^{\dagger}$	$35.06^{\dagger}$	30.99^{**}
	P-star Model					
0.30	25.54^{**}	23.16^{*}	20.69	$30.98^{\dagger}$	16.94	7.29
0.40	28.36^{**}	14.32	14.24	$33.97^{\dagger}$	15.04	10.19
0.50	24.25^{*}	16.09	17.07	$29.31^{\dagger}$	11.56	11.60
0.60	22.09	17.45	17.50	$33.64^{\dagger}$	16.42	18.50
0.70	17.19	14.97	12.00	$40.28^{\dagger}$	19.78	17.49
	AR(3) Model					
0.30	28.69^{**}	24.58	15.39	26.45^{**}	24.64^{*}	17.25
0.40	$33.02^{\dagger}$	26.02^{**}	15.87	$31.52^{\dagger}$	24.63^{*}	17.53
0.50	28.10^{**}	15.97	14.73	24.74^{*}	25.36^{**}	28.77^{**}
0.60	19.47	19.50	17.73	25.85^{**}	$38.47^{\dagger}$	$43.79^{\dagger}$
0.70	15.73	17.25	19.25	$32.10^{\dagger}$	$43.53^{\dagger}$	$50.04^{\dagger}$
	AR(1) Model					
0.30	$19.33^{\dagger}$	$23.76^{\dagger}$	$24.89^{\dagger}$	$31.72^{\dagger}$	$37.69^{\dagger}$	$53.56^{\dagger}$
0.40	$18.99^{\dagger}$	$27.39^{\dagger}$	$26.28^{\dagger}$	$31.33^{\dagger}$	$38.89^{\dagger}$	$53.60^{\dagger}$
0.50	$23.92^{\dagger}$	$30.76^{\dagger}$	$32.16^{\dagger}$	$29.96^{\dagger}$	$36.88^{\dagger}$	$54.05^{\dagger}$
0.60	$18.18^{\dagger}$	$24.34^{\dagger}$	$22.05^{\dagger}$	$30.14^{\dagger}$	$37.48^{\dagger}$	$53.61^{\dagger}$
0.70	$17.07^{\dagger}$	$18.32^{\dagger}$	$24.79^{\dagger}$	$30.34^{\dagger}$	$38.18^{\dagger}$	$53.52^{\dagger}$

note: 1. *, **, † mean that the test rejects parameter stability at 10%, 5%, and 1% significant levels, respectively.
2. the lag lengths are set by AIC.

Based on $B^*(\Omega)$ test result, we find evidence that Phillips curve are still unstable even in the ara of great moderation. This result coincides with Atkeson and Ohanian

TABLE 2. Test Results: Mean/Variance Stability

Model	1982-2007	1982-1989	1991-2007
Phillips Curve(BPC)	30.53^*	20.72	25.22
P-star Model	44.11^*	44.33^*	36.53
AR(1)	$25.42^\dagger$	13.02^*	$32.84^\dagger$
AR(3)	24.31	22.17	20.01
Unconditional Vairance	$42.96^\dagger$	$15.45^\dagger$	$38.77^\dagger$

(2001) and Stock and Watson (2006)'s findings that the forecasting power of backward looking type of inflation models is not improved even after 1990's because of instability.

Another finding is that the test does not have identical results across quantiles and it depends on the sample periods. $B(\Omega)$ has a tendency to strongly reject the null hypothesis for parameters of lower quantile levels, while it accept it for higher quantile levels in periods 1962-current, and 1982-current. In 1991-current period, $B^*(\Omega)$ rejects stability in upper quantile levels and accept it in lower quantiles. It has similar test result in AR(3) models. Consequently, the test supports the asymmetry of the inflation response to economic shocks. It also implies that the inflation density forecast methods based on parametric distribution such as inflation fan charts may lose the accuracy of the forecast because, as noted above, assumed stability of the shape of the distribution contradicts the test result.

For P^* models, both $B(\Omega)$ and $B^*(\Omega)$ reject the stability in whole sample period (1962-2007), But they could not reject in more recent subsample periods. However, it should be cautious to admit the forecasting power of P^* model. The main problem of the forecasting power P^* model is the weak relationship between P^* gap and the inflation. But accepting the stability of the model does not necessarily mean that the model shows the close relationship.

The AR(1) model is shown to be most unstable. Both tests reject the stability at 1% significant level. For the AR model based on AIC lag decision (AR(3)), $B(\Omega)$ could not reject the stability in 1991-2007. However, $B^*(\Omega)$ shows evidence of instability in lower quantile while it has opposite test results in upper quantiles.

In summary, I find evidence of instability in all models for the whole sample periods (1962-2007). The evidence of instability after 1990's depends on the selected model. For some models, we find different test result across different quantile levels.

6. Conclusion

This paper applies the optimal tests for parameter instability to linear conditional quantile models both in parametric and semiparametric setups. Tests functions of Lee (2008) are shown to be applicable. Tick-exponential distribution allows the parametric test function to be asymptotically valid even under misspecified underlying distribution.

The application of the tests to quantile models for the U.S. inflation process shows evidence of instability of various quantile models. The quantile models are still rejected to be stable even after 1990's, while the conditional mean is shown to be stable in some of the inflation models such as Phillips Curve. The nonidentity of instability test results implies that inadequacy of the density forecast based on parametric density function as well as the asymmetric response of inflation to various economic shocks.

7. Proofs

7.A. **Proof of Lemma ??.** Let $\Delta\dot{a}_t = \dot{a}_t(\bar{\beta}+T^{-1/2}\delta)-\dot{a}_t(\bar{\beta})$ and $\Delta 1_t = 1_{[y_t<X_t\bar{\beta}+T^{-1/2}X_t\delta]}-1_{[y_t<X_t b\bar{e}ta]}$. The score function can be written as

$$\dot{\ell}_t(\bar{\beta}+T^{-1/2}\delta) = \dot{\ell}_t(\bar{\beta}) + \Delta\dot{a}_t\dot{\ell}_t(\bar{\beta}) - \dot{a}_t(\bar{\beta})X_t\Delta 1_t + \Delta\dot{a}_t X_t \Delta 1_t \tag{7.1}$$

$$\begin{aligned} \Rightarrow T^{-1/2}\sum_{t=1}^{[sT]}\dot{\ell}_t(\bar{\beta}+T^{-1/2}\delta) \quad &= \quad T^{-1/2}\sum_{t=1}^{[sT]}\dot{\ell}_t(\bar{\beta}) + T^{-1/2}\sum_{t=1}^{[sT]}\Delta\dot{a}_t\dot{\ell}_t(\bar{\beta}) - \\ & \quad T^{-1/2}\sum_{t=1}^{[sT]}\dot{a}_t(\bar{\beta})X_t\Delta 1_t + T^{-1/2}\sum_{t=1}^{[sT]}\Delta\dot{a}_t X_t\Delta 1_t \end{aligned} \tag{7.2}$$

Following two Taylor expansions are used to prove the lemma.

$$\begin{aligned} \Delta\dot{a}_t \quad &= \quad T^{-1/2}\ddot{a}_t(\bar{\beta})X_t'\delta + o_p(\sqrt{T}) \\ F(T^{-1/2}X_t'\delta) - F(0) \quad &= \quad T^{-1/2}X_t'\delta f(0) + o_p(\sqrt{T}) \end{aligned} \tag{7.3}$$

By using (7.3), (7.2) can be rewritten as,

(7.4)

$$T^{-1/2}\sum_{t=1}^{[sT]}\dot{\ell}_t(\bar{\beta}+T^{-1/2}\delta) = T^{-1/2}\sum_{t=1}^{[sT]}\dot{\ell}_t(\bar{\beta}) + T^{-1/2}\sum_{t=1}^{[sT]}\ddot{a}_t X_t \dot{\ell}_t(\bar{\beta})'\delta - sK(\bar{\beta})\delta + o_p(1)$$

The second term converges to zero because

$$\begin{aligned} T^{-1}\sum E\left[\ddot{a}_t(\bar{\beta})X_t\dot{\ell}_t(\bar{\beta})\right] &= T^{-1}\sum E\left[(\alpha - 1_{[\epsilon_t<0]})\ddot{a}_t(\bar{\beta})X_t X_t'\dot{\ell}_t(\bar{\beta})\right] \\ &= 0 \end{aligned}$$

which completes the proof. $\diamond$

7.B. **Proof of Lemma ??.** By Lemma 4) in Lee (2008), Lemma **??** is proved if $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}\dot{\ell}_t^* = sW(s)$ where $W(s)$ is a Wiener process. Lemma A.2 of Komunjer (2005) shows that $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}\dot{\ell}_t$ satisfies the conditions to apply the CLT for $\alpha-$mixing sequence. But by Condition **??** (iv) and the condition that ϵ_t is iid, $\{X_t f(\epsilon_t)\}$ is globally covariance stationary with long-run covariance $E[X_t X_t' f(\epsilon_t)^2]$. Therefore, it satisfies the conditions for the FCLT for $\alpha-$mixing sequence (theorem 7.30 in White (2001)), which completes the proof. $\diamond$

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