University of Connecticut

Department of Economics Working Paper Series

Bootstrap Tests for Structural Breaks When the Regressors and Error Term are Nonstationary

Dong Jin Lee
University of Connecticut

Working Paper 2011-05

March 2011

---

341 Mansfield Road, Unit 1063
Storrs, CT 06269–1063
Phone: (860) 486–3022
Fax: (860) 486–4463
http://www.econ.uconn.edu/

Abstract

This paper considers tests for structural breaks in linear models when the regressors and the serially dependent error process are unstable. The set of models contains various economic circumstances such as the structural breaks in the regressors and/or the error variance, and a linear trend model with I(0)/I(1) error. We show that the existing heteroscedasticity robust tests and the fixed regressor bootstrap method of Hansen (2000) have severe size distortion problem even in the asymptotics. We suggest a method which combines the fixed regressor bootstrap and the sieve-wild bootstrap method to nonparametrically approximate the

the serially dependent unstable error process. The suggested method is shown to asymptotically replicates the true distribution of the existing tests under various circumstances. Monte Carlo experiments show significant improvements both in the size and the power properties. Once the size is controlled by the bootstrap, Wald type tests have better power properties relative to LM type tests.

Journal of Economic Literature Classification: C10, C12, C22

Keywords:structural break, sieve bootstrap, fixed regressor bootstrap, robust test, break in linear trend

1. **Introduction**

Structural instability is a common problem in macroeconomic and finance model dealing with time series data. A change in economic policy or market conditions may cause the adjustment of the behavior of economic agents, thereby changing the economic relationship. As a result, parameter instability has always been an important concern in econometric modeling and research has developed many powerful tests for structural break such as Andrews (1993)'s supF, Andrews and Ploberger (1994)'s expF, and Elliott and Müller (2006)'s qLL.

One critical assumption in the existing structural break tests is that the regressor and the error term are stable, so that the break in the coefficient is the only source of instability in the model. However, there are many circumstances in practice under which the assumption is not valid. For example, if one suspects that a macroeconomic shock causes a break of the relationship between inflation and the output in the Phillips curve, it is reasonable to suspect that the shock may also impacts on the output process, causing its mean or volatility shifts. In addition, many macroeconomic and financial variable display breaks in the unconditional volatility which implies breaks in the unconditional variance of the error term. (van Dijk, Osborn, and Sensier (2002), Stock and Watson (1999)) When the assumption is violated, the existing tests suffer from the size distortion problem and the test results are strongly depends on the choice of tests. Even within the supF or expF tests, the results are different between using Wald type or LM type tests for F.

Hansen (2000) tackles the problem arisen from the unstable regressors using fixed regressor bootstrap method. His method still does not explicitly consider the break in unconditional variance in the error process. Furthermore, this method is restrictive when the error term is serially correlated. Fixed regressor still suffers from serious size distortion problem because the bootstra data eliminates the serial dependency of the errors. Adding lagged dependent variables causes poor finite sample powers against moderate or big sized breaks (Vogelsang (1999)), and has a misspecification problem as well which is a common motivation to use HAC estimator rather than adding lags. And his method can be shown to be valid aainst a local structural break in the error variance, although he assumes the variance is stable by assuming that the average of the square of the error process is constant. But this method still has a size distortion problem when the error process is serially u correlated which is common in econometric modeling.

Another popular example of an unstable error process is to test structural break in the linear trend model when the error term can be either I(0) or I(1) but unknown. Perron and Yabu (2009), Vogelsang (2001), and Saygnsoy and Vogelsang (2004) exploit the existing tests to obtain testing methods that are robust to the order of integration of the error term. Under certain circumstances, however, Perron and Yabu (2009)'s method have correct size only if the error term is normally distributed even in large samples. Vogelsang (2001), and Saygnsoy and Vogelsang (2004) randomize the existing tests which result in the loss of power.

The main contribution to this paper is to propose an asymptotically correct sized method by using the mixture of Hansen (2000)'s fixed regressor bootstrap and Sieve bootstrap methods. The method does not require the knowledge of the form of serial correlation in the error process. The suggested bootstrap method asymptotically replicates the true distribution of the existing tests under various feasible nonstandard circumstances such as the serially correlated error term with the break in the variance, together with the structural breaks in the marginal distribution of regressors. Finite sample Monte Carlo study shows that the method is far superior to the existing methods such as fixed regressor bootstrap and the test using HAC estimator.

Sieve bootstrap method, first proposed by Bühlmann (1997), is to approximate an infinite-dimensional, nonparametric time series process by a sequence of finite dimensional autoregressive models in which the order is increasing as $T \to \infty$. The sieve approximations are valid for all data generating processes that belong to the stationary linear AR(∞) class. This method is widely used in various standard and nonstandard testing problem such as unit root tests (Psaradakis (2003), Chang and Park (2003)), tests for fractional integration and cointegration (Gerolimetto and Procidano (2008), Poskitt (2006)), and tests on long-run average parameter in panel data models (Fuertes (2007)).

Rather than using *iid* normal to generate the fixed regressor bootstrap sample, we propose a sample drawing from a AR(p) approximation of the residual of the regression in the model with single structural break. The error process in the AR(p) is generated from the *iid* standard normal random variable multiplied by the residual of the original equation.

This two step bootstrap method is shown to asymptotically mimic the true serially correlated error process with possible breaks in the variance process. Accordingly, the tests have correct asymptotic sizes under various unstable regressors and the error processes with a wide range of serial dependencies. Sieve bootstrap is valid for autoregressive model in both I(0) and I(1).(Inoue and Kilian (2002)) Consequently, the proposed method can be applied to the trend testing with possible unit root error process. Monte Carlos study shows that the suggested method is superior the existing methods in both size and power properties.

This paper is organized as follows: Section 2 analyzes the circumstances under which the existing tests fail to obtain correct asymptotic sizes. Section 3 Introduces the bootstrap method. Section 4 applies the suggested method to test structural breaks in the linear trend model. Section 5 performs Monte Carlo studies. And Section 6 concludes.

2. Existing Structural Break Tests Under Unstable Regressors and the Error term

Consider a linear regression model

$$y_t = x_t'(\beta_0 + \beta_t) + u_t \quad t = 1, \ldots, T \tag{2.1}$$

where y_t is the dependent variable, x_t is the $k \times 1$ vector of regressors, β_0 and β_t are the $k \times 1$ vectors of coefficients, and u_t is the error term with mean zero. The mean zero condition can be replaced by the quantile restriction that $Pr(u_t < 0) = \alpha$ if one is interested in linear quantile model. We are interested in testing single structural break of the linear regression coefficients in which the null and the alternative hypotheses are defined as

$$H_0 : \beta_t = \quad 0 \text{ for all } t = 1, \ldots, T$$
$$H_1 : \beta_t = \begin{pmatrix} 0 & \text{for } t \leq \tau T \\ \bar{\beta} & \text{for } \tau T < t \leq T \end{pmatrix}$$

Structural break tests such as Andrews (1993), Andrews and Ploberger (1994), and Elliott and Müller (2006) assume that $\Sigma_m^{-1/2} \sum_{t=1}^{[sT]} X_t u_t$ is asymptotically described by the multivariate Wiener process, where Σ_m is the covariance matrix of $\Sigma_m^{-1/2} \sum_{t=1}^{[sT]} X_t u_t$, and $[a]$ stands for the smallest integer less than equal to a. It implies that the process of x_t and the variance of u_t do not show any instabilities including structural breaks.

The purpose of this paper is to propose a method that is robust to a set of models which capture a wide range of possible circumstances in macroeconomic and financial analysis including the possible breaks in x_t and u_t process. To this aim, we first define the general serial correlation and heteroscedastic property of u_t in the following condition.

Condition 1. *(1) $u_t = \varphi(L)\epsilon_t$ where $\varphi(L)$ is the lag operator polynomial, $\epsilon_t = \sigma_t\eta_t$, η_t is iid $(0,1)$, and σ_t is strictly positive and bounded in probability.*

(2) σ_t is independent of η_τ for all $0 < t, \tau \leq T$.

(3) $\varphi(z) \neq 0$ for all $|z| \leq 1$ and $\sum_{j=1}^{\infty} j|\varphi_j| < \infty$.

Condition 1 is the standard conditions for the functional central limit theorem of serially correlated series u_t except the variance σ_t^2 is time varying. It is general enough to allow for many weakly dependent stationary process (σ_t =constant), and nonstationary process relying on the shape of σ_t process. σ_t^2 can be the square of *iid* which implies the conventional heteroscedastic process, or it can represent structural break in which $\sigma_t = \sigma_0$ for $1 \geq t \leq \pi T$ for $0 < \pi < 1$, and $\sigma_0 + \sigma_1$ otherwise. Detailed property of σ_t is described in Condition 2. Condition 1.(3) implies that the process does not include unit root case. Consequently, the instability of the unconditional variance σ_t is the only possible source of the unstable u_t. However, next section 4 shows that the extension to the unit root error process case is straightforward so that (2.1) covers a variety of linear economic models. I consider the unit root case separately because unit root error term presents only in the model with deterministic regressors such as linear trend ones. Otherwise (2.1) would be spurious regression model which is not the interest of this paper. The second term in Condition 1.(3)

requires that the autoregressive coefficients $\{\varphi_j\}$ shrinks faster than under typical absolute summability condition. But this faster shrinkage is necessary in order for the sieve approximation to asymptotically capture the serially correlated process.

This paper considers possible instabilities of σ_t and the marginal distribution of x_t. We define the instabilities based on their asymptotic properties in the sense that the purpose of the test is to suggest asymptotically robust testing methods. Rather than imposing a specific form of asymptotics which will restrict the processes, we give the condition that the partial sum of the joint process of x_t and σ_t weakly converses to a certain bounded function as Condition 2.

Condition 2.

(1) $$\frac{1}{T}\sum_{t=1}^{[sT]} x_t x_t' \Rightarrow M(s)$$

(2) $$\frac{1}{T}\sum_{t=1}^{[sT]} x_t x_{t+i}' \sigma_{t-j}^2 \Rightarrow \Gamma(s,i,j) \text{ for all } i = 1,\ldots, j = 1,\ldots$$

(3) $$\frac{1}{T}\sum_{t=1}^{[sT]} \sigma_t^2 \Rightarrow S(s)$$

where $[z]$ is the largest integer less than or equal to z and $\Rightarrow$ implies weak convergency.

Condition 2 (1) is equivalent to Hansen (2000) and covers a wide range of possible unstable process of x_t such as linear trend ($M(s) = s^2/3$) and cointegration ($M(s) = \int_0^s W_x(r)^2 dr$, See Hansen (2000) for details). It also accounts for various types of structural breaks in the marginal distribution of x_t. For example, if x_t contains any macroeconomic variables such as growth and inflation, and the data set covers 1980s. Then we would expect a shift in volatility of x_t since the era of great moderation. Accordingly, the regressor x_t would have the form; $x_t = \Lambda_1 Z_t$ before great moderation, and $x_t = \Lambda_2 Z_t$, after great moderation where Λ_1 and Λ_2 are diagonal matrices, and Z_t is weakly stationary with covariance matrix Σ_Z. It implies that $M(s) = \Sigma_Z + (s - \tau)(\Lambda_1 - \Lambda_2)\Sigma_Z 1_{[s>\tau]}$ where τT is the time of structural break and $1_{[]}$ is the indicator function.

Condition 2.(2) is required in order for the variance of $\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} x_t u_t$ to weakly converge. Note that $\Gamma(s, i, j)$ is nonlinear in s and is possibly random. It implies that heteroscedasticity and autocorrelation consistent (HAC) estimation method do not necessarily provide the consistence estimators for the asymptotic variance. As Kiefer, Vogelsang, and Bunzel (2000) noted, HAC estimators require that $\sum_t \sum_\tau E[x_t u_t] E[X_\tau u_\tau]' \to sV$ for a deterministic V. Consequently, the existing tests using HAC estimator will not obtain correct size property even in the asymptotics.

Condition 2.(3) defines weak convergency of the variance process. It is broad enough to cover a various heteroscedastic processes both stable and unstable. For example, it represents a typical heteroscedastic process or random coefficient model if $S(s)$ is linear in s, and captures (multiple) structural breaks if $S(s)$ is a step function. Since ϵ_t is uncorrelated with $\{x_t\}$, this condition also captures various stationary and nonstationary processes of conditional heteroscedasticity such as GARCH, T-GARCH, and integrated GARCH. In case that the conditional heteroscedasticity is integrated, $S(s) = \int_0^s \sigma^2(r)dr$ where $\sigma^2(r)$ is the positive (non)random process. However, it does not cover standard GARCH process in which the unconditional variance is constant while the conditional variance is time varying. Various stochastic volatility process such as integrated autoregressive stochastic volatility are also captured in the condition. Condition 2.(3) is similar to Cavaliere and Taylor (2008) but is weaker in that $S(s)$ can be random and the heteroscedasticity is clearly related with $\{x_t\}$ which is reasonable in practice.

This paper considers three popular and powerful tests for a single structural break; Andrews (1993), Andrews and Ploberger (1994), and Elliott and Müller (2006). Hansen (2000)'s wild bootstrap method to derive the critical values of the three tests are also considered. Andrews (1993)(supT) and Andrews and Ploberger (1994)(ExpT) 's tests are defined as

$$SupT = \sup_{\lambda_0 < s < 1-\lambda_0} T_T(s) \tag{2.2}$$

$$ExpT = \int_{\lambda_1}^{1-\lambda_1} exp\left[\frac{1}{2}T_T(s)\right] ds \tag{2.3}$$

where λ_0, and λ_1 are trimming parameters, $T_T(s)$ stands for Chow test functions for a break at a given break time sT, which can be either Wald type test or LM type test. Wald test is defined as

$$W_T(s) = T(R\tilde{\beta})'(R\widetilde{\Sigma}_\beta R')^{-1}(R\tilde{\beta}) \tag{2.4}$$

where $R = (0_k, I_k)$, 0_k is the $k \times k$ matrix of zeroes, I_k is the $k \times k$ identity matrix, $\tilde{\beta}$ is the estimator of (β, β_t) in equation 2.1 with a break point sT, and $\widetilde{\Sigma}_\beta$ is the covariance estimator of $\tilde{\beta}$. LM test is defined as

$$LM_T(s) = \frac{m_T(s)' m_T(s)}{s(1-s)} \tag{2.5}$$

$$m_T(s) = \widehat{\Sigma}_m^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{sT} x_t \tilde{u}_t$$

where $\tilde{u}_t$ is the regression residual of the equation 2.1 under the null hypothesis of no structural break ($\beta_t = 0$), and $\widehat{\Sigma}_m$ is the estimator of the covariance of $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t u_t$. Under homoscedasticity, Σ_m can be estimated by $\hat{\sigma}^2 \frac{1}{T}\sum_{t=1}^{T} x_t x_t'$, $\hat{\sigma}^2 \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2$, while HAC estimators should be applied under heteroscedasticity and serial correlation.

Elliott and Müller (2006)'s test (qLL) is the increasing function of the maximum invariance transformed likelihood function with normal distribution which is defined as

$$LR_T = \int \exp\left[\sigma^{-2}\left(\sum_{t=1}^{T} x_t'\beta_t\tilde{u}_t - \frac{1}{2}\sum_{t=1}^{T} x_t'\beta_t\beta_t'x_t\right)\right] d\nu_\beta$$

where ν_β is the probability measure of the random walk process with Gaussian error. Note that qLL does not explicitly consider a single structural break but focus on a random walk coefficient breaks. However, they show that the test has asymptotically optimal power against structural break processes as long as the number of breaks increases as the sample size T goes to infinity, and show that qLL has similar small sample powers with $SupT$ and $ExpT$ against a single break.

Asymptotic critical values of the tests are calculated based on the assumptions that $\frac{1}{T}\sum_{t=1}^{[sT]} x_t x_t' \rightarrow sE[x_t x_t']$ in probability and $\Sigma_m^{-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t u_t \Rightarrow W(s)$ where $W(s)$ is the standard multivariate Wiener process. Note that these assumptions are even stricter than the typical assumptions for the consistency of OLS estimator because the convergency requires uniformly in s for both terms. Hansen (2000)'s fixed regressor bootstrap provides asymptotically correct critical values when the condition for $\frac{1}{T}\sum_{t=1}^{[sT]} x_t x_t'$ is mitigated to Condition 2.(1). The idea is that as long as $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t u_t$ is asymptotically described by the Normal distribution, its asymptotic distribution is equivalent to that of $\hat{\sigma}_u\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t\eta_t$ where η_t is the iid standard normal and $\hat{\sigma}_u$ is the estimator of the standard deviation of u_t. Similar idea can be applied to heteroscedastic errors in which $\hat{\sigma}_u\eta_t$ is replaced by $\hat{u}_t\eta_t$, where $\hat{u}_t$ is the residual from the regression of the equation 2.1 with structural break. Consequently, the bootstrap samples y_t^b and x_t^b are generated from $y_t^b = \eta_t$ and $x_t^b = x_t$. This method is valid only if the covariance of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t\hat{u}_t$ is consistently estimated by $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} \hat{u}_t^2 x_t x_t'$. However, a small deviation from this condition such as the nonstationary unconditional variance process and/or serial correlation of u_t may lead to a serious size distortion even in the fixed regressor bootstrap method. The following proposition shows that both conventional tests and the fixed regressor bootstrap tests do not provide correct size property under the above conditions.

Theorem 1. *Let's define $B_1(V_1(s))$ and $B_2(V_2(s))$ be the multivariate normal distribution with conditional variance $V_1(s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_j\psi_{j+i}\Gamma(s^*, |j-i|, i)$ and $V_2(s) = \sum_{j=1}^{\infty}\varphi_j^2 V(s,0,j)$, respectively where $s^* = s - \frac{j-i}{T}$. Under Condition 1 and 2,*

(1) The asymptotic distributions of the conventional tests are

$$\sup W_T \Longrightarrow \sup_{\pi_1 s<1-\pi_1} W, \qquad \sup LM_T \Longrightarrow \sup_{\pi_1 s<1-\pi_1} LM$$

$$\exp W_T \Longrightarrow ln\left[\frac{1}{1-2\pi_2}\int \exp\left(\frac{1}{2}W\right)dr\right], \exp LM_T \Longrightarrow ln\left[\frac{1}{1-2\pi_2}\int \exp\left(\frac{1}{2}LM\right)dr\right]$$

$$qll \Longrightarrow \sum_{i=1}^{k}[a_iJ_{1,i}(1)^2 + a_i^2\int_0^1 J_{1,i}(s)^2ds + \frac{2a_i}{1-e^{-2a_i}}\times$$

$$\{e^{-a_i}J_{1,i}(1) + a_i\int_0^1 e^{-a_is}J_{1,i}(s)ds\}^2 - \{J_{1,i}(1) + a_i\int J^i(s)ds\}^2]$$

$$W = B_1^*(V_1(s))'M(s)^*B_1^*(V_1(s))$$

$$LM = B_1^*(V_1(s))'\Sigma(1,1)^{-1}B_1^*(V_1(s))$$

(2) the asymptotic null distribution of fixed regressor bootstrap tests are

$$\sup F \Longrightarrow \sup_{\pi_1 s < 1-\pi_1} B_2^*(V_2(s))' M(s)^* B_2^*(V_2(s))$$

$$\exp LM \Longrightarrow ln \left[\frac{1}{1-2\pi_2} \int \exp\left(\frac{1}{2} B_2^*(V_2(s))' M(s)^* B_2^*(V_2(s)) \right) dr \right]$$

$$qll \Longrightarrow \sum_{i=1}^{k} [a_i J_{2,i}(1)^2 + a_i^2 \int_0^1 J_{2,i}(s)^2 ds + \frac{2a_i}{1-e^{-2a_i}} \times$$
$$\{e^{-a_i} J_{2,i}(1) + a_i \int_0^1 e^{-a_i s} J_{2,i}(s) ds\}^2 - \{J_{2,i}(1) + a_i \int J_{2,i}(s) ds\}^2]$$

where $B_i^*(s) = B_i(V_i(s)) - M(s)M(1)^{-1}B_i(V_i(1))$, $M_1^*(s) = \Sigma(s,s) + M(s)M(1)^{-1} \times \Sigma(1,1)M(1)^{-1}M(s) - 2\Sigma(s,1)M(1)^{-1}M(s)$, $M_2^*(s) = \Sigma_0 - M_x(s)M_x(1)^{-1}\Sigma_0 M_x(1)^{-1}M_x(s)$, $J_{i,j}(s) = B_{i,j}^{**}(s) - \int_0^s e^{\lambda - s}[B_{i,j}^{**}(\lambda)]d\lambda$ *and* $B_{i,j}^{**}(s)$ *is the* j^{th} *element of* $V_i(1)^{-1}B_1^*(s)$, $\Sigma(i,j)$ *is the asymptotic counterpart of any covariance estimator* $\hat{\Sigma}(i,j)$ *of* $E\left[\frac{1}{T}(\sum_{t=1}^{[iT]} x_t u_t)(\sum_{\tau=1}^{[jT]} x_\tau u_\tau)\right]$, *and* $\Sigma_0 = E[\frac{1}{T}\sum_{i=1}^{T} u_t^2 x_t x_t']$.

$B_i(V(s))$ would be the variance transformed Brownian motion if $V_i(s)$ are deterministic. The asymptotics of HAC estimator of $\hat{\Sigma}_{(i,j)}$ is not specified in Theorem 1 because the convergence generally depends on the estimator and the specific model. $\hat{\Sigma}_{(i,j)}$ generally does not obtain the consistency in two reasons. First, the estimator may not have enough samples to obtain the consistency. For example, if one uses the truncation $\pi = 0.05$ in either $supW$ or $expW$ with 100 samples, then we need to estimate the autocovariance of order p using just $6-p$ observations in order to compute $W_T(\pi + 1/T)$. Second, Condition 2 is broader than the conventional condition for the consistency of HAC estimator so that the model covers many nonstandard circumstances in which HAC cannot be applied. Even though $\hat{\Sigma}_{(i,j)}$ is consistent, the asymptotic distribution of $W_T(s)$ and $LM_T(s)$ are still not the square of the Brownian bridge, .e. $[B(s) - sB(1)]'[B(s) - sB(1)]$ under which the asymtptotic critical values are calculated. Consequently, the suggested critical values in $supT$, $expT$, and qLL are not valid in our general circumstances. Hansen (2000)'s fixed regressor bootstrap also fails to asymptotically replicates the true distribution of the tests. If White standard error is applied to the estimator $\hat{\Sigma}_{(i,j)}$, then $M_1^*(s) = M_2^*(s)$. But the distribution of $W_T^b(s)$ $(LM_T^b(s))$ is still different from that of $W_T(s)(LM_T(s))$ because of the discrepancy of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t u_t$ and $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t u_t^b$. Therefore if unstable x_t and σ_t are present together with serially correlated error process, neither conventional structural break test, nor fixed regressor bootstrap obtain the asymptotically correct size properties. Note that even within sup or exp tests, the asymptotic distribution is different between W_T and LM_T. Consequently we expect that the test results generally depend on the choice between LM_T and W_T.

3. Sieve Wild Bootstrap Tests

The previous section shows that the instability in the regressor and the error term affect the asymptotic distribution of the existing tests in complicated ways so that using traditional tests would mislead the actual structural break of the model. In this section, I propose an advanced method by combining Sieve bootstrap with the wild bootstrap. The basic idea of the sieve bootstrap, first proposed by Bühlmann (1997), is to approximate an infinite-dimensional, nonparametric model by a sequence of finite dimensional autoregressive models in which the order is increasing as $T \to \infty$. Accordingly, the serial dependence of the data is removed by an finite autoregressive approximation. Bühlmann (1997), Psaradakis (2003) Gerolimetto and Procidano (2008), Poskitt (2006), Chang (2004), Chang, Park, and Song (2006), and Inoue and

Kilian (2002) show that asymptotic validity of the sieve bootstrap in various time series models.

The model in this paper is different from the typical setup of the sieve bootstrap in three ways; first, the sieve approximation is done by not using the original process u_t but using the regression residuals of the model 2.1 as a proxy for u_t. Consequently, the suggested bootstrap generate the data from the regression.

$$\tilde{u}_t = \sum_{i=1}^{p(T)} \phi_i^* \tilde{u}_{t-i} + \epsilon_t^*$$

where $\tilde{u}_t$ is the OLS residual of the equation 2.1 with a single structural break where the break point τ can be estimated by Bai (1996)'s method. Note that $\tilde{u}_t = u_t - Z_t'(\frac{1}{T}\sum_{t=1}^{T} Z_t Z_t')^{-1}\frac{1}{T}\sum_{t=1}^{T} Z_t u_t$ where $Z_t = (X_t', X_t' 1_{[t>\hat{\tau}T]})'$. Since the second term is still $op(1)$ even under Condition 2, using $\tilde{u}_t$ does not affect the consistency while it changes the asymptotic distribution of $\{\hat{\phi}_i\}$. Our bootstrap does not required the asymptotic distribution. Thus replacing u_t by $\hat{u}_t$ does not change the asymptotic property of the bootstrap distribution.

Second, u_t is not covariance stationary in that the unconditional variance σ_t is nonstationary and time varying. The aim of the bootstrap is to generate sample error series u_t^b that asymptotically satisfy Condition 2, i.e. $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} X_t u_t^b$ weakly converges to $B(V_1(s))$ and the bootstrap data is able to construct $\widehat{M}^*(s)$ which also weakly converges to $M^*(s)$. To do this, it is required that $\hat{\phi}_i$ converges in probability to ϕ_i and $|\hat{\epsilon}_t - \epsilon_t|$ is asymptotically negligible. Bühlmann (1997) uses Yule-Walker equation to obtain the consistency of $\{\hat{\phi}_i\}$. But if u_t is not covariance stationary as Condition 2, the typical Yule-Walker equation does not exist. In order to tackle the problem, we show that the average of the autocovariance function $\frac{1}{T}\sum_{t=1}^{T} E[u_t u_{t-j}]$

provides the Yule-Walker type equation in the asymptotics, and that this asymptotic version leads to the same conclusion as Bühlmann (1997).

Third, like Hansen (2000), the bootstrap resampling is down by using the external distribution, rather than by using the empirical distribution of the bootstrap residuals. Consequently, the suggested bootstrap is the mixture of a wild bootstrap and a sieve bootstrap. The asymptotic validity of wild bootstrap under nonstationary unconditional volatility is proved under various circumstances. (**?**Cavaliere and Taylor (2008), Xu (2008)) This paper shows the possible extension of the existing work into our nonstandard framework. The suggested bootstrap approximations are constructed in the following manner.

(1) Estimate the model and the time of the break τ, $\hat{\tau} = \arg_{\tau} \min \sum_{t=1}^{T} \hat{u}_{t,\tau}^2$ where $\hat{u}_{t,\tau}^2$ is the OLS residuals for a specific τ. Save $\{\hat{u}_{t,\tau}\}$

(2) Estimate the $AR(p)$ approximation of $\hat{u}_t$

$$\hat{u}_t = \phi_1 \hat{u}_{t-1} + \phi_2 \hat{u}_{t-2} + \ldots + \phi_p \hat{u}_{t-p} + \epsilon_t$$

where the lag p is determined by AIC.

(3) Generate the bootstrap residual process $\{u_t * (b)\}$ such that

$$u_t(b) = \hat{\phi}_1 u_{t-1}(b) + \hat{\phi}_2 u_{t-2}(b) + \ldots + \hat{\phi}_p u_{t-p}(b) + \epsilon_t(b)$$

where $\epsilon_t(b) = \hat{\epsilon}_t \eta_t$, $\hat{\epsilon}_t$ is the OLS residual of step 2), $(p\hat{h}i_1, \ldots, p\hat{h}i_p)$ are the OLS estimator, and $\eta_t \sim N(0,1)$

(4) set $y_t(b) = u_t(b)$. Regress $y_t(b)$ on x_t under the null hypothesis and construct the test function, $SupF_i^b$ as defined in

(5) Iterate 1)-3) B times to get the empirical distribution of bootstrap test function $SupF^b$. Calculate the critical-value of $SupF^b$ using bootstrap distribution.

Let$T(Z)$ and $T(Z^b)$ denote the original test function and its bootstrap counterpart, respectively. $T(\cdot)$ can be either $SupF$, $ExpLM$ or qLL. The main contribution in this paper is that the bootstrap test $T(Z^b)$ weakly converges in probability to the limiting counterpart of $T(Z)$ no matter what x_t and ϵ_t are stable or not, which is shown in the following theorem.

Theorem 2. *Under Condition 1 and 2,*

$$Sup_Z \left| Pr\left[T(Z^b) < t\right] - Pr\left[T(Z) < t\right] \right| = o_p(1) \tag{3.1}$$

4. **Structural Break in linear trend**

Previous section does not cover the case where u_t is I(1) because in a typical set-up I(1) error implies spurious regression. This section considers an another case with unstable error process; a linear deterministic trend in which the noise term is either stationary or have an unit root. Precisely the data generating process is considered to be

$$y_t = (\alpha + \delta t) + (\alpha_1 + \delta_1 t)1_{[t>\tau T]} + u_t \tag{4.1}$$

$$\phi(L)u_t = \epsilon_t \quad \epsilon_t \sim iid(0, \sigma^2) \tag{4.2}$$

The autoregressive process of u_t represents either stationary or nonstationary, i.e. the roots of the lag operator polynomial $\phi(z)$ would either be all outside the unit circle or have a one. We may consider a break in level only ($\alpha_t = \bar{\alpha}$ for $t > \tau T$, and $\delta_t = 0$), or a break in trend only ($\delta_t = \bar{\alpha}$ for $t > \tau T$, and $\alpha_t = 0$), or both. This type of model is of practical importance in macroeconomic and finance analysis. For example, if y_t is $\log GDP$, equation 4.1 is to detect a break in the economic growth rate. However it is often the case that there is no prior knowledge about whether the noise term is I(0) or I(1). If u_t is I(1) but the test function is based on equation 4.1, the asymptotic critical value is totally different from those the existing ones are provided because $\frac{1}{T}\sum u_t$ is not asymptotically a Brownian bridge. Instead, if one uses a first-differenced data, then the test would be largely misleading if the true u_t is I(1). Consequently, dealing with the order of the integration is crucial in structural break testing.

Vogelsang (2001), Saygnsoy and Vogelsang (2004), and Perron and Yabu (2009) are around this problem using different approaches. Vogelsang (2001) keeps equation 4.1 in constructing tests but modified the existing tests function in order to obtain the unified critical values both in I(0) and I(1). The test function is defined as

$$supW_{ur} = \sup_{\lambda_0 < s < 1-\lambda_0} W_{T,s} exp\left[-bJ_T^*\right] \tag{4.3}$$

where $W_{T,s}$ is as defined in 2.4, $J_T^* = inf_{\lambda_0<\tau<1-\lambda_0} J_T$, and J_T is an unit root test statistic for u_t. One candidate of $J_{T,\tau}$ is Park (1990) which is

$$J_{T,\tau} = \frac{RSS_R - RSS_U}{RSS_U} \tag{4.4}$$

where SSR_R is the residual sum of squares from the estimation of 4.1 with a break at $[\tau T]$, and SSR_U is the residual sum of squares from the estimation of the following unrestricted model

$$y_t = (\alpha + \delta t) + (\alpha_1 + \delta_1 t)1_{[t>\tau T]} + \sum_{i=k1}^{9} \pi_i t^i + u_t^* \tag{4.5}$$

Similar test using Andrews and Ploberger (1994) can be defined by $ExpW_{UR} = \int_{\lambda_1}^{1-\lambda_1} exp\left[\frac{1}{2}W_T(s)exp\left(-bJ_T^*\right)\right] ds$. J_T converges to zeros for all τ if u_t is I(0), and thus $exp[bJ_T^*]$ converges to one so that the test function is asymptotically equivalent to the standard $supW$ test. If u_t is I(1), J_T has nondegenerate limiting distribution and the asymptotic distribution of $supW_{ur}$ is different from that of $supW$ both under H_0 and H_1. Vogelsang (2001), and Saygnsoy and Vogelsang (2004) choose the value of b in which the critical value of $supW_{ur}$ is identical for both I(0) and I(1) error process. Consequently, the test has correct asymptotic size regardless of the order of the integration in u_t. $supW_{ur}$ has the same asymptotic power as $supW$ if u_t is I(0). However, the test has poor power properties if u_t is I(1) because, by construction, $supW_{ur}$ which is $supW$ multiplied by a random function $exp[bJ_T^*]$ is considered to be a randomized version of $supW$ test, which in general loses testing power.

Instead of equation 4.1, Perron and Yabu (2009) exploit a GLS set-up as follows.

$$(1-\hat{\phi}L)y_t = (1-\hat{\phi}L)\left[(\alpha_0 + \delta_0 t) + (\alpha_1 + \delta_1 t)1_{[t>\tau T]}\right] + (1-\hat{\phi})u_t \tag{4.6}$$

where $\hat{\phi}$ is from the estimation of the equation $\hat{u}_t = \phi\hat{u}_{t-1} + \sum_{i=1}^{p} \varphi\Delta u_{t-i} + \epsilon_t^*$. Then as long as $\hat{\phi}$ is consistent for both $I(0)$ and $I(1)$ error process, $(1-\hat{\phi})u_t$ is I(0) regardless of the order of the original u_t, and the conventional $supW$ and $ExpW$ tests can be applied to the model. Perron and Yabu (2009) use a superefficient estimator of ϕ, which is defined as

$$\hat{\phi} = \begin{cases} \hat{\phi}_{OLS} & \text{if } |\hat{\phi}_{OLS}| < 1 - T^{1/2} \\ 1 & \text{otherwise} \end{cases} \tag{4.7}$$

where ϕ_{OLS} is the OLS estimator. This method is expected to have better powers than Vogelsang (2001)'s because it avoids the use of unit root tests which generally causes additional type II error. However, it has nontrivial size distortion problem if u_t is I(1). their Wald and LM tests in (2.5) and (2.4) are asymptotically the square of the Brownian bridge, under I(1) u_t only if ϵ_t is normally distributed. The reason is that if $\hat{\phi} = 1$, equation (4.6) is

$$\Delta y_t = \delta_0 + \alpha_1 1_{[t=\tau T]} + \delta_1 1_{[t>\tau T]} + \Delta u_t \tag{4.8}$$

Then the test statistic contains the term $\sum_{t=1}^{[sT]} 1_{[t=\tau T]}\hat{u}_t = \hat{u}_{[\tau T]}$ and the asymptotic distribution of the test depends on the true distribution of a single observation $hatu_t$ of which asymptotic theories cannot be applied. Intuitively, the model (4.8) rarely recognizes the difference between one time temporary change in the constant and an occurrence of an outlier in Δu_t. Perron and Yabu (2009) eliminate the possibility of the existence of outliers by assuming that ϵ_t is Gaussian and the test regards any outliers as structural breaks. This problem would not be serious if one is interested in testing only a shift in the constant ($\delta_1 = 0$) because indeed such an outliers can act as a level shift. However, if one is testing structural break in both the constant term and the trend, any single occurrence of an outlier would largely mislead that the trend has a structural break, and the test rarely accept the null of no break if the error distribution has fat tails. This problem is significant in that excess kurtosis often have better explanatory power for variables in macroeconomics and finance. Another problem of Perron and Yabu (2009) is that the test cannot be applied to $supW_T$ test because its asymptotic distribution is still different between I(0) and I(1) although one uses (4.6). Hence the method cannot be used to detect the time of a structural break. Consequently, Vogelsang (2001) dominates Perron and Yabu (2009) in terms of size property, while Perron and Yabu (2009) is superior in power. And the choice between the two test functions is a matter of preference.

The point is whether one can replicate the true distribution of the test statistics without randomizing the existing powerful tests. As is shown in the previous section, the asymptotic distribution of $supW$, $ExpLM$ and qLL depends on the asymptotics of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}\hat{u}_t$ and $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}t\hat{u}_t$, which are $B(s)-sB(1)$, and $B(s)-sB(1)-\inf_0^s B(r)dr+s\int_0^1 rB(r)dr$ under I(0), and $\int_o^s W(r)dr-\int_0^1 rW(1)dr$ and , $\int_o^s rW(r)dr-s\int_0^1 rW(1)dr$ respectively. If for simplicity $\sum_{i=1}^{\infty}\varphi\Delta u_{t-i}=0$, $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}\hat{u}_t^b$ and $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}t\hat{u}_t^b$ would have the same asymptotic distribution as the above where $u_t^b=\hat{\phi}u_{t-1}^b+\epsilon_t$ and ϵ_t is iid normal. Thus, we can replicate the asymptotic distribution of the existing tests. In this section, we show that the bootstrap method suggested in the previous section replicates those partial sum processes. The following theorem shows that the Sieve Wild type bootstrap is asymptotically valid in testing breaks in linear trend.

Theorem 3. *Suppose the model follows* (4.1), (4.2)*. Then* (3.1) *in Theorem 2 holds for any* $supT_T$*,* $expT_T$ *and* qLL*.*

The power function is asymptotically equivalent to that of Vogelsang (2001) under I(0) becuase the test function is asymptotically identical, but it is superior under I(1) in that it avoids the randomizing by not using the additional random term $\exp[bJ_{T,\tau}]$. The suggested method has better size property than that of Perron and Yabu (2009) because it is asymptotically correct-sized regardless of the distribution of ϵ_t. The asymptotic powers are not identical because our method uses equation (4.1) and Perron and Yabu (2009) use (4.6). The asymptotic power depends on the asymptotics of $\frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]}x_t\hat{u}_t$. For example, under I(1), it is $B(s)-sB(1)-[s(1-s)-max(s-\tau,0]\delta$ in Perron and Yabu (2009) and $\int_0^s rB(r)dr-s^3\int_0^1 rB(r)dr-[\frac{s^3(1-\tau^3)}{3}-max(\frac{s^3-\tau^3}{3},0)]\delta$ for a break at time $[\tau T]$ with size of $T^{-1/2}\delta$. Figure 1 shows the asymptotic powers of Perron and Yabu (2009) and the bootstrap method. The test function using the original equation (4.1) has asymptotically better power than that using the GLS setup (4.6), which provides another motivation to use (4.1).

5. **Comparative Simulation Study**

This section examines the performance of the bootstrap method in finite samples through Monte Carlo experiments. We study a simple linear regression model and a linear deterministic trend model, respectively.

5.1. **linear regression model.** Consider

$$y_t = (\alpha + \alpha_t) + x_t(\beta_0 + \beta_t) + u_t \quad t = 1, \ldots, T \tag{5.1}$$

$$u_t = \rho u_{t-1} + \epsilon_t \tag{5.2}$$

where y_t and x_t are scalars, $\epsilon_t = \sigma_t \eta_t$, and $\eta_t \sim iidN(0,1)$. The AR coefficient ρ is set to 0.7. We examine various plausible circumstances under which x_t and/or ϵ_t are unstable and simulate the empirical sizes and the powers of the tests. The circumstances include;

(1) Single structural break in the mean of x_t and the variance of u_t, i.e.

$$x_t = \begin{cases} z_t, & \text{if } t < \tau_0 T, \\ \mu_x + z_t, & \text{otherwise.} \end{cases} \qquad \sigma_t = \begin{cases} \sigma_0, & \text{if } t < \tau_0 T, \\ \sigma_1, & \text{otherwise.} \end{cases}$$

where z_t is covariance stationary.

(2) Single structural break in the variance of x_t and the variance of u_t, i.e.

$$x_t = \begin{cases} \sigma_x^1 z_t, & \text{if } t < \tau_0 T, \\ \sigma_x^2 z_t, & \text{otherwise.} \end{cases} \qquad \sigma_t = \begin{cases} \sigma_0, & \text{if } t < \tau_0 T, \\ \sigma_1, & \text{otherwise.} \end{cases}$$

(3) Break in σ_t every t, i.e.

$$x_t = z_t, \qquad \sigma_t = |\sum_{i=1}^{t} \iota_i|$$

where $\iota_i \sim iidN(0,1)$.

$\{z_t\}$ is generated from AR(1) model with *iid* Gaussian error. In case (1), the mean of x_t shifts from one to three and σ_t jumps from one to two at $\tau_0 = 0.5T$. In case (2), the both standard deviation and σ_t shift from one to two at the same break point as case (1). We consider two different sample sizes, T=100, and 200. 2,000 replications are generated for each T, and 1,000 samples are drawn in order to calculate the bootstrap critical values. We first estimate the empirical size at 5% critical level. Five different test functions are considered: $supW$, $supLM$, $ExpW$, $ExpLM$ and qLL. The sizes are compared with the conventional tests with HAC estimators of variance, and Hansen's fixed regressor bootstrap.

The results are presented in Table 2. For all three cases, classic tests of $supW$ and $ExpW$ based on asymptotic critical values, have empirical sizes more than 80%, which means that the test has more chance to detect a structural break although the truth is no break. Fixed regressor bootstrap method is better than the classical method, but the size distortion is still serious in most of the cases. Although fixed regressor bootstrap tackles the break in x_t, the size distortion by disregarding serially correlated error process is severe so that the method can hardly applied to such circumstances. The suggested Sieve Wild bootstrap method is far superior to the other two methods.

But the method does not provide a complete remedy in that the small sample with T=100, it still has an over size problem. LM type tests and qLL have far better size properties than Wald type.

As a next step, I calculate small sample powers of the test and compare the existing tests. We do not compare the powers across different method because once the size is adjusted, those tests are identical. The size adjusted small sample powers are shown at figure 2, and 3. The figures show that LR types test have better powers than qLL and LM. LM and Wald types have opposite properties. LM tests have better size while they suffer from poor powers. On the other hand, Wald tests have severe size distortion problem, but the power is superior. qLL has the best size based on asymptotic critical values. The powers are moderately better than LM tests but is inferior to Wald tests. In conclusion, qLL is recommended if one wants to keep using the conventional tests with asymptotic critical values. If one uses the suggested bootstrap, supW or expW would be far better.

5.2. **linear deterministic detrend.** Consider

$$y_t = \alpha + \delta t + u_t$$

$$u_t = \rho u_{t-1} + \eta_t \qquad \eta_t \sim iid\ Laplace$$

In order to examine the effect of non Gaussian error term on Perron and Yabu (2009), η_t is generated from an *iid* Laplace distribution. Various levels of ρ is considered such as $\rho = 1, 0.95, 0.9, 0.8$). Like the previous simulation, we draw 2,000 samples and the bootstrap critical values are calculated from 2,000 iterations. Table 2 shows the result of the empirical sizes when $T = 200$. For all level of ρs, the suggested bootstrap method has sizes close to the actual size 5% in each of 5 tests. PY has serious size distortion problem when ρ is either one or close to one. The sizes are almost 90%, which implies that outliers make the test most likely detect a structural break even though the true trend function is stable. Considering that majority of time series in macroeconomics and finance has fatter tails than normal, PY is hardly applied to test for a break in both the constant and the trend. Sizes of $SupW_{UR}$ and $ExpW_{UR}$ are close to 5%, while the test is relatively conservative as ρ gets closer to one.

Figure 4 and 5 show the size adjusted powers of the tests. Among three methods, our bootstrap method provides the best small sample powers. While $supW_{UR}$ has good size properties, its power is the worst. Our bootstrap dominates the other two both in sizes and powers.

6. **Conclusion**

Structural breaks in the marginal distribution of regressors and the error variance are common in macroeconomic and finance modeling. When the error process are serially correlated, existing methods of either using asymptotic critical values and the fixed regressor bootstrap have severe size distortion problem. This problem partly explains why in many literatures such as Giacomini and Rossi (2006), different types of tests yield contradictory results. Consequently, dealing with these instability is crucial in testing structural breaks.

In this paper, we get around the problem by approximating the true distribution of the unstable error process using the mixture of sieve and wild bootstraps. Combining the bootstraps with Hansen (2000)'s fixed regressor bootstrap method captures the instabilities both in the regressors and the error variance, thereby provides the correct asymptotic critical values of sup, exp, and qLL tests.

The suggested sieve-wild bootstrap is applied to testing structural breaks in linear trend models in which the order of the integration in the error term is unknown *a priori*. The test obtains correct asymptotic critical values and have superior powers to the existing tests.

Monte carlo simulation strongly supports the use of suggested methods. Once the size is controlled by the bootstrap, one need to use Wald type test rather than LM.

Appendix A. Proofs

A.1. **Proof of Theorem 1.** (i) Proof of (1): In order to prove the asymptotic distribution of $supT$ and $ExpT$, we have only to show that

$$W_T(s) \Longrightarrow B_1^*(V_1(s))' M(s)^* B_1^*(V_1(s))$$
$$LM_T(s) \Longrightarrow B_1^*(V_1(s))' \Sigma_{1,1}^{-1} B_1^*(V_1(s))$$

by continuous mapping theorem.

(a) Convergence of $W_T(s)$

Under H_0, $W_T(s)$ can be rewritten as

(A.1)
$$W_T(s) = S^*(s)'[\hat{\Sigma}(s,s)+M_x(s)M_x(1)^{-1}\hat{\Sigma}(1,1)M_x(1)^{-1}M_x(s)-2\hat{\Sigma}(s,1)M_x(1)^{-1}M_x(s)]S^*(s)$$

where $S^*(s) = S(s) - M_x(s)M_x(1)^{-1}S(1)$, $S(s) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t u_t$, $M_X(s) = \frac{1}{T}\sum_{t=1}^{sT} x_t x_t'$, and $\hat{\Sigma}(i,j)$ is the estimator of $E\left[\frac{1}{T}\left(\sum_{t=1}^{[iT]} x_t u_t)(\sum_{\tau=1}^{[jT]} x_\tau u_\tau)'\right)\right]$. If $\hat{\Sigma}(i,j) \Longrightarrow \Sigma(i,j)$, then by Condition 2)

$$\begin{aligned}
&\hat{\Sigma}(s,s) + M_x(s)M_x(1)^{-1}\hat{\Sigma}(1,1)M_x(1)^{-1}M_x(s) - 2\hat{\Sigma}(s,1)M_x(1)^{-1}M_x(s) \Longrightarrow \\
&\Sigma(s,s) + M(s)M(1)^{-1}\Sigma(1,1)M(1)^{-1}M(s) - 2\Sigma(s,1)M(1)^{-1}M(s)
\end{aligned} \tag{A.2}$$

Since $\frac{1}{\sqrt{T}}\sum_{t=1}^{sT} X_t\epsilon_{t-i}$ is asymptotically Normal with conditional Variance $\Gamma(s,o,j)$, $\frac{1}{\sqrt{T}}\sum_{t=1}^{sT} X_t u_t = \sum_{i=1}^{\infty}\varphi_i \frac{1}{\sqrt{T}}\sum_{t=1}^{sT} x_t\epsilon_{t-i}$ is also asymptotically normal with the variance is the asymptotic counterpart of $\frac{1}{T}(\sum_{t=1}^{sT} X_t u_t)(\sum_{t=1}^{sT} X_t u_t)'$.

$$\frac{1}{T}(\sum_{t=1}^{sT} X_t u_t)(\sum_{t=1}^{sT} X_t u_t)' = \frac{1}{T}(\sum_{i=1}^{\infty}\varphi_i \sum_{t=1}^{sT} x_t \epsilon_{t-i})(\sum_{j=1}^{\infty}\varphi_j \sum_{t=1}^{sT} x_t \epsilon_{t-j})$$
$$= \frac{1}{T}\sum_{t=1}^{[sT]}\sum_{\tau=1}^{[sT]}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_t x_\tau' \varphi_i \varphi_j \epsilon_{t-i}\epsilon_{\tau-j}$$
$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varphi_i\varphi_j \frac{1}{T}\sum_{t=1}^{[sT]}\sum_{\tau=1}^{[sT]} x_t x_\tau' \epsilon_{t-i}\epsilon_{\tau-j} \tag{A.3}$$

Since η_t is iid, any cross terms($t-i \neq \tau - j$) converge to zero. Consequently,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\varphi_i\varphi_j \frac{1}{T}\sum_{t=1}^{[sT]}\sum_{\tau=1}^{[sT]} x_t x_\tau' \epsilon_{t-i}\epsilon_{\tau-j} = \sum_{i=0}^{\infty}\varphi_i^2 \frac{1}{T}\sum_{t=1}^{[} sT] x_t x_t' \epsilon_{t-i}^2 \tag{A.4}$$
$$= \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\varphi_i\varphi_j \frac{1}{T}\sum_{t=1}^{[sT-|j-i|]} x_t x_{t+|j-i|}' \epsilon_{t-i}^2 + o_p(1)$$
$$\Longrightarrow \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\varphi_i\varphi_j \Gamma(s^*, |j-i|, i) = V_1(s)$$

Consequently by A.4 and Condition 2

$$S^*(s) = B_1(V_1(s)) - M(s)M(1)^{-1}B_1(V_1(1)) \tag{A.5}$$

Combining A.2 and A.5 completes the proof of $W_T(s)$ convergency.

(b) Convergence of $LM_T(s)$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{sT} x_t \hat{u}_t = \frac{1}{\sqrt{T}}\sum_{t=1}^{sT} x_t u_t - \frac{1}{T}\sum_{t=1}^{sT} x_t x_t' \left(\frac{1}{T}\sum_{t=1}^{T} x_i x_i'\right)^{-1} \frac{1}{\sqrt{T}}\sum_{j=1}^{[sT]} x_j u_j = S^*(s)) \tag{A.6}$$

under H_0. Consequently

$$LM_T(s) = S^*(s)'\hat{\Sigma}(1,1)^{-1}S^*(s) \Longrightarrow B_1(V_1(s))'\Sigma(1,1)^{-1}B_1(V_1(s)) \tag{A.7}$$

(c)Convergence of qLL: (A.7) also implies the convergency of qLL by Lemma 6 of Elliott and Müller (2006).

(ii) Proof of (2): Let u_t^b be the fixed regressor bootstrap sample, i.e. $u_t^b = \tilde{u}_t \eta_t$ where $\tilde{u}_t$ is the residual of the estimation of the unrestricted model with a structural break and η_t is the iid normally distributed random variable. Accordingly, we will prove the convergence of $W_T^b(s)$ only. Using heteroscedasticy version, Wald statisic is defined as

$$W_T^b(s) = S^{b*}(s)' \left[\hat{\Sigma}_0 - M_x(s)M_x(1)^{-1}\hat{\Sigma}_0 M_x(1)^{-1}M_x(s)\right] S^{b*}(s) \tag{A.8}$$

where $S^{b*}(s)$ equivalent to $S^*(s)$ with u_t replaced by u_t^b, and $\hat{\Sigma}_0 = \frac{1}{T}\sum_{t=1}^T \hat{u}_t^2 x_t x_t'$. Since

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^T u_t^2 x_t x_t' = \sum_{i=0}^\infty \varphi_i^2 \frac{1}{T}\sum_{t=1}^T \sigma_{t-i}^2 x_t x_t' \eta_t + o_p(1) \\ \Longrightarrow \sum_{i=0}^\infty \varphi_i^2 \Gamma(s,0,i) \end{aligned} \tag{A.9}$$

we would get

$$\begin{aligned} \frac{1}{\sqrt{T}}\sum_{t=1}^{[sT]} x_t u_t^b &\Longrightarrow W_2(V_2(s))' \\ \hat{\Sigma}_0 &\Longrightarrow V_2(1) \end{aligned} \tag{A.10}$$

which completes the proof.

Lemma 1. *Let* $\bar{\gamma}_t(j) = \frac{1}{T}\sum_{t=1}^{T-j}\gamma_t(j)$ *and* $\hat{\gamma}(j) = \frac{1}{T}\sum_{t=j+1}^T \hat{u}_t\hat{u}_{t-j}$*. Under condition 1 and 2, for any* $\delta > 0$*, and* $p(T) \le T^{r/(2(r-2)}$*,* $r \ge 4$

$$\max_{0 \ge j \ge p(T)} |\hat{\gamma}(j) - \bar{\gamma}_t(j)| = o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}] \tag{A.11}$$

Outline of the Proof This is an extension of An, Chen, and Hannan (1982) Theorem 1 in which u_t is covariance stationary. $\hat{\gamma}(j) - \bar{\gamma}_t(j)$ can be rewritten as

$$\hat{\gamma}(j) - \bar{\gamma}_t(j) = \frac{1}{T}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\psi_m\psi_l\sum_{t=1}^{T-j}[\epsilon_{t-l}\epsilon_{t+j-m} - \sigma_{t-l}^2 I_{[l=m-j]}] \tag{A.12}$$

Since $[\epsilon_{t-l}\epsilon_{t+j-m} - \sigma_{t-l}^2 I_{[l=m-j]}]$ is square integrable martingale. The proof in **?** can be applied to this case. Hence we skip the detailed proof.

Lemma 2. *Under condition 1 and 2,*

$$\max_{0\geq j\geq p(T)} |\hat{\varphi}_j - \varphi_{T,j}| = o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}] \tag{A.13}$$

Proof) Using $\hat{\gamma}(j) = \sum_{i=1}^{p(T)} \hat{\varphi}_i\hat{\gamma}(i-j)$, we can write

(A.14)

$$\sum_{i=1}^{p(T)}(\hat{\varphi}_i - \varphi_{T,i})\hat{\gamma}(i-j) = -\sum_{i=0}^{p(T)}\varphi_i\hat{\gamma}(i-j) + \sum_{i=1}^{p(T)}(\varphi_i - \varphi_{T,i})(\hat{\gamma}(i-j) - \bar{\gamma}_t(i-j)) + \sum_{i=0}^{p(T)}\varphi_i\bar{\gamma}_t(i-j)$$

The second term in the left hand side is $o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}]$ from lemma 1 and equation (A.30). Let's define $a(T)$ which is $p(T) < a(T)$ and $a(T)/T \to 0$. The third term in the left hand side can be rewritten as

$$\begin{aligned}\sum_{i=0}^{p(T)}\varphi_i\bar{\gamma}_t(i-j) &= -\sum_{i=p(T)+1}^{\infty}\varphi_i\bar{\gamma}_t(i-j) = -\sum_{i=p(T)+1}^{a(T)}\varphi_i\bar{\gamma}_t(i-j) - \sum_{i=a(T)+1}^{\infty}\varphi_i\bar{\gamma}_t(i-j)\\ &= -\sum_{i=p(T)+1}^{a(T)}\varphi_i\hat{\gamma}(i-j) - \sum_{i=a(T)+1}^{\infty}\varphi_i\bar{\gamma}_t(i-j) - \sum_{i=p(T)+1}^{a(T)}\varphi_i(\bar{\gamma}_t(i-j) - \hat{\gamma}(i-j))\end{aligned}$$

(A.15)

$$= -\sum_{i=p(T)+1}^{a(T)}\varphi_i\hat{\gamma}(i-j) + o_p(a(T)^{-2}) - o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}]\sum_{i=p(T)+1}^{a(T)}\varphi_i$$

The $o_p(a(T)^{-2})$ term is because

$$a(T)^2 \sum_{i=a(T)+1}^{\infty} \varphi_{i+j}\varphi_i \frac{1}{T} \sum_{t=j+1}^{T} \sigma_{t-j}^2 \leq \sum_{i=a(T)+1}^{\infty} |a(T)\varphi_I|^2 \bar{\sigma}^2 \to 0$$

by Condition 1 (iii). Thus the left hand side of (A.14) is

$$-\sum_{i=0}^{a(T)} \varphi_i \hat{\gamma}(i-j) + o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}] \sum_{i=p(T)+1}^{a(T)} \varphi_i + o_p(a(T)^{-2}) \tag{A.16}$$

Using (A.23)

$$\begin{aligned}
\sum_{i=0}^{a(T)} \varphi_i \hat{\gamma}(i-j) &= \sum_{i=0}^{a(T)} \varphi_i(\hat{\gamma}(i-j) - \bar{\gamma}_t(i-j)) + \sum_{i=a(T)+1}^{\infty} \varphi_i \bar{\gamma}_t(i-j) \\
&= o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}] \sum_{i=0}^{a(T)} \varphi_i + o_p(a(T)^{-2}) \\
&= o_p[T^{-1/2}\{p(T)lnT\}^{2/r}(ln\ lnT)^{2(1+\delta)/r}] \qquad \text{(A.17)}
\end{aligned}$$

which completes the proof.

A.2. **Proof of Theorem 2.** Here we will prove the Wald type test only. The case of LM types and qLL is pretty similar to the Wald case. From Theorem 1, the bootstrap test function $T^b(\cdot)$ weakly converges to $T(\cdot)$ if $\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \Rightarrow B_1(V_1(s))$ and $\frac{1}{T} \sum_{i=1}^{[sT]} u_t^{b2} X_t X_t' \Rightarrow \Sigma_0$. By equation (A.3), the weak convergency of the first one can be verified if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \hat{\varphi}_i\hat{\varphi}_j \frac{1}{T} \sum_{t=1}^{[sT]}\sum_{\tau=1}^{[sT]} x_t x_\tau' \hat{\epsilon}_{t-i}\hat{\epsilon}_{\tau-j} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varphi_i\varphi_j \frac{1}{T} \sum_{t=1}^{[sT]}\sum_{\tau=1}^{[sT]} x_t x_\tau' \epsilon_{t-i}\epsilon_{\tau-j} \tag{A.18}$$

converges to zero in probability, and the convergence of the second one follows the first convergency. Thus it is sufficient to show that

$$\sum_{i=0}^{\infty} |\hat{\varphi}_i - \varphi_i| = o_p(1) \tag{A.19}$$

$$\hat{\epsilon}_t = \epsilon_t + o_p(1) \tag{A.20}$$

i) Proof of A.19

$$\sum_{j=0}^{\infty} j|\hat{\varphi}_j - \varphi_j| < \sum_{j=0}^{p(T)} j|\hat{\varphi}_j - \varphi_T, j| + \sum_{j=0}^{p(T)} j|\varphi_{T,j} - \varphi_j| + \sum_{j=p(T)+1}^{\infty} j|\varphi_j| \tag{A.21}$$

$$= I + II + III \tag{A.22}$$

I is $o_p(1)$ by lemma 2 letting $p(T) = o((\frac{\ln T \ln\ln T}{T})^{1/2})$. In order to prove **II**, we generate a stationary data that have the same asymptotic autocovariance structure as u_t. $u_t = \sum_{j=1}^{\infty} \varphi_j u_{t-j} + \epsilon_t$ implies that

$$\gamma_t(j) = \sum_{j=1}^{\infty} \varphi_j \gamma_{t-j}(i-j) \tag{A.23}$$

Now taking average for all t yields

$$\bar{\gamma}_t(j) = \sum_{i=1}^{\infty} \varphi_j \bar{\gamma}_{t-j}(i-j) \tag{A.24}$$

Let's define $a(T)$ such that $a(T) \to \infty$, $p(T) < a(T)$, and $\frac{a(T)}{T} \to 0$. Let $\bar{\gamma}_T(j) = \sum_{i=1}^{\infty} \varphi_{i+j}\varphi_i \frac{1}{T}\sum_{t=1}^{T} \sigma_t^2$. Since $\bar{\gamma}_T(j) = \sum_{i=1}^{\infty} \varphi_{i+j}\varphi_i \frac{1}{T}\sum_{t=j+1}^{T} \sigma_{t-j}^2$, we can write

$$\bar{\gamma}_t(j) = \bar{\gamma}_T(j) + \sum_{i=1}^{\infty} \varphi_{i+j}\varphi_i \frac{j}{T} O_p(1)$$

Thus equation (A.24) can be rewritten as

$$\bar{\gamma}_T(j) = \sum_{i=1}^{a(T)} \varphi_j \bar{\gamma}_T(i-j) + \sum_{i=a(T)+1}^{\infty} \varphi_j \bar{\gamma}_{t-j}(i-j) + R_t \tag{A.25}$$

where $R_t = \sum_{k=1}^{a(T)} \varphi_k \sum_{i=0}^{\infty} \varphi_i \varphi_{i+j} \frac{\max[i,j]}{T} O_p(1)$ which is $O_p(T^{-1})$ by Condition 1(iii). Let's define $\gamma^*(i) = \sum_{j=1}^{a(T)} \varphi_j \gamma^*(i-j)$ where $\gamma^*(i) = \sum_{i=1}^{a(T)} \varphi_i \varphi_{i+j} \bar{\sigma}^2$ and $\bar{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} \sigma_t^2$. Note that $\gamma^*(j)$ does not depend on t and is related to $\bar{\gamma}_T(j)$ as below

$$\gamma^*(j) - \bar{\gamma}_T(j) = \sum_{i=1}^{a(T)} \varphi_i \left(\gamma^*(i-j) - \bar{\gamma}_T(i-j)\right) + \sum_{i=a(T)+1}^{\infty} \varphi(i)\gamma(i-j) \tag{A.26}$$

The second term of the left hand side can be rewritten as $\sum_{j=a(T)+1}^{\infty} \varphi_j \sum_{i=1}^{\infty} \varphi_{j+i}\varphi_i \bar{\sigma}^2(1-j/T)$ which is $o_p(a(T)^{-2})$ Thus

$$max|\gamma^*(i)-\bar{\gamma}_T(i)| \leq max|\gamma^*(i)-\bar{\gamma}(i)| \sum_{j=1}^{a(T)} \varphi_j + o_p(a(T)^{-2}) \leq max|\gamma^*(i)-\gamma(i)|\Delta + o_p(a(T)^{-2})$$

for some $0 < \Delta < \infty$, which implies that $|\gamma(i)^* - \bar{\gamma}_T(i)|$ is $o_p(a(T)^{-2})$, and thereby $|\gamma(i)^* - \bar{\gamma}(i)|$ is $o_p(a(T)^{-2})$ for $0 < i < a(T)$. Proof can be done by generating a covariance stationary data which has the same autocovariance function $\{\gamma^*(i)\}$, i.e. let's generate a data $\{u_t^*\}$ from the equation

$$u_t^* = \sum_{i=1}^{a(T)} \varphi_i u_{t-i}^* + \epsilon_t^* \tag{A.27}$$

where $\epsilon_t^* = \bar{\sigma}(1)\eta_t$ and η_t is $iid(0,1)$. Then we can define the spectral density of u_t^* denoted by $f(w)$. Let $(\varphi_{T,1}^*, \ldots, \varphi_{T,p(T)}^*)$ be the linear projection coefficient for u_t^* using the first $p(T)$ lags. Then

$$\gamma^*(k) = \sum_{j=1}^{p(T)} \gamma^*(k-j)\varphi_{T,j}^* = \sum_{j=1}^{p(T)} \varphi_{T,j}^* \bar{\sigma}(1)^{-2} \int_{-\pi}^{\pi} f(w)e^{-i(k-j)w}dw$$

$$\gamma^*(k) = \sum_{j=1}^{a(T)} \gamma^* k - j\varphi_j = \sum_{j=1}^{a(T)} \varphi_j \bar{\sigma}(1)^{-2} \int_{-\pi}^{\pi} f(w)e^{-i(k-j)w}dw \tag{A.28}$$

Consequently,

$$\sum_{j=1}^{p(T)} \bar{\sigma}(1)^{-2}(\varphi_{T,j}^* - \varphi_j) \int_{-\pi}^{\pi} f(w)e^{-i(k-j)w}dw = \sum_{j=a(T)+1}^{\infty} \gamma_{k-j}\varphi_j\bar{\sigma}(1)^{-2} \int_{-\pi}^{\pi} f(w)e^{-i(k-j)w}dw$$

Now applying Baxter's inequality yields

$$\sum_{j=1}^{p(T)} j|\varphi_{T,j}^* - \varphi_j| < \sum_{j=p(T)+1}^{\infty} j|\varphi_j| \to 0 \tag{A.29}$$

by Condition 2,(iii). Therefore,

$$\sum_{i=1}^{p(T)} i|\varphi_{T,i} - \varphi_i| \leq \sum_{i=1}^{p(T)} i|\varphi^*_{T,i} - \varphi_i| \sum_{i=1}^{p(T)} i|\varphi_{T,i} - \varphi^*_{T,i}| \tag{A.30}$$

The first term of the left hand side converges to zero from (A.29). The second term also converges to zero because

$$\sum_{i=1}^{p(T)} |\varphi_{T,i} - \varphi^*_{T,i}|\bar{\gamma}_T(j-i) \leq max|\gamma(j-i) - \gamma^*(j-i)| + o_p(a(T)^{-2}) \sum_{i=1}^{p(T)} \varphi_j = o_p(a(T)^{-2}) \tag{A.31}$$

which completes the proof of the zero convergency of II.

III converges to zero by Condition 1.(iii) which completes the proof of A.19

ii) Proof of A.20

$$\hat{u}_{t,\tau} = u_t - X_t(\frac{1}{T}\sum_{i=1}^{T} X_i X_i')^{-1} \frac{1}{T}\sum_{i=1}^{T} X_i \epsilon_i = u_t - o_p(T^{-1/2}) \tag{A.32}$$

$$\begin{aligned}
\hat{\epsilon}_t &= \hat{\phi}(L)\hat{u}_{t,\tau} \\
&= \hat{\phi}(L)u_t + \hat{\phi}(L)X_t(\frac{1}{T}\sum_{i=1}^{T} X_i X_i')^{-1} \frac{1}{T}\sum_{i=1}^{T} X_i \epsilon_i \\
&= \epsilon_t + P_t + Q_t + R_t
\end{aligned} \tag{A.33}$$

$$\text{where } P_t = \sum_{j=0}^{p} \hat{\phi}_j X_{t-j} (\frac{1}{T}\sum_{i=1}^{T} X_i X_i')^{-1} \frac{1}{T}\sum_{i=1}^{T} X_i \epsilon_i = u_t - o_p(T^{-1/2})$$

$$Q_t = \sum_{j=0}^{p} (\hat{\varphi}_j - \varphi_{T,j}) u_{t-j}$$

$$R_t = \sum_{j=0}^{p} (\varphi_{T,j} - \varphi_j) u_{t-j}$$

Q_t and R_t converge to zero by lemma 2, which completes the proof.

A.3. **Proof of Theorem 3.** The proof is similar to that of Theorem 2, i.e it is sufficient to show that

$$\sum_{i=0}^{\infty} |\hat{\phi}_i - \phi_i| = o_p(1) \tag{A.34}$$

$$\frac{1}{T}\sum_{t=1}^{T} \hat{\epsilon}_t^2 = \sigma^2 + o_p(1) \tag{A.35}$$

(A.34) is shown by Palm, Smeekes, and Urbain (2006) and replacing $\hat{\phi}_{OLS}$ by $\hat{\phi}$ results in the same convergency which is shown by Perron and Yabu (2005). As is the proof of Theorem 2, the proof of the second convergence follows from the first one. Consequently, we will skip the detailed proof.

References

An, H., Z. Chen, and E. Hannan (1982): "Autocorrelation, autoregression and autoregressive approximation," *The Annals of Statistics*, 10(3).

Andrews, D. (1993): "Tests for Parameter Instability and structural Change with Unknown Change Point," *Econometrica*, 61(3).

Andrews, D., and W. Ploberger (1994): "Optimal Tests When a Nuisance Parameter Is Present Only Under the Alternative," *Econometrica*, 62(3).

Bai, J. (1996): "Testing for Parameter Constancy in Linear Regressions: An Empirical Distributiojn Function Approach," *Econometrica*, 64(3).

Bühlmann, P. (1997): "Sieve Bootstrap for Time series," *Bernouilli*, 3.

Cavaliere, G., and A. R. Taylor (2008): "Bootstrap unit root tests for time series with nonstationary volatility," *Econometric Theory*, 24.

Chang, Y. (2004): "Bootstrap Unit Root Tests in Panels with Cross-sectional dependency," *Journal of Econometrics*, 120.

Chang, Y., and Y. Park (2003): "A seive bootstrap for the test of a unit root," *Journal of Time Series Analysis*, 24.

Chang, Y., Y. Park, and K. Song (2006): "Bootstrappiung cointergrating regressions," *Journal of Econometrics*, 133.

Elliott, G., and U. Müller (2006): "Efficients Tests for General Persistent Time Variation in Regression Coefficients," *Review of Economics Studies*, 73, 907–940.

Fuertes, A.-M. (2007): "Sieve bootstrap t-test on long-run average parameters," *Computational Statistics and Data Analysis*, 52, 3354–3370.

Gerolimetto, M., and I. Procidano (2008): "A test for fractional cointegration using the sieve bootstrap," *Statistical Methods and Application*, 17, 373–391.

Giacomini, R., and B. Rossi (2006): "How stable is the forecasting performance of the yield curve for output growth? Oxford Bulletin of Economics and Statistics," *Oxford Bulletin of Economics and Statistics*, 68.

Hansen, B. (2000): "Testing for Structural Change in Conditional Models," *Journal of Econometrics*, 97(2).

Inoue, A., and L. Kilian (2002): "Bootstrapping autoregressive processes with possible unit roots," *Econometrica*, 70(1).

Kiefer, N., T. Vogelsang, and H. Bunzel (2000): "Simple robust testing of regression hypothesis," *Econometrica*, 68, 695–714.

Palm, F. C., S. Smeekes, and J.-P. Urbain (2006): "Bootstrap Unit Root Tests: Comparison and Extensions," .

Park, J. (1990): "Testing for Unit Roots and Cointegration by Variable Addition," *Advances in Econometrics: Cointegration, Spurious Regressions and Unit Roots, ed. by T. Fomby and F. Rhodes.*, pp. 107–134.

Perron, P., and T. Yabu (2005): "Estimating Deterministic Trends with an Integrated or Stationary Noise Component," *Unpublished manuscript, Boston Univ.*

——— (2009): "Testing for Shifts in Trend With an Integrated or Stationary Noise Component," *Journal of Business and Economic Statistics*, 27(3), 369–396.

Poskitt, D. (2006): "Properties of the sieve bootstrap for fractionally integrated and non-invertible processes," *Journal of Time Series Analysis*, 29(2), 224–250.

Psaradakis, H. (2003): "A Seive Bootstrap for Stationarity," *Statiatistics and Probability Letter*, 62, 263–274.

Saygnsoy, z., and T. Vogelsang (2004): "Powerful Tests of Structural Change That are Robust to Strong Serial Correlation," *unpublished manuscript.*

Stock, J., and M. Watson (1999): "A comparison of linear and nonlinear univariate models for forecasting macroeconomic time series," in *Cointegration, Causality and Forecasting: A Festchrift in Honour of Clive W.J. Granger*, pp. 1–44. Oxford University Press.

van Dijk, D., D. Osborn, and M. Sensier (2002): "Changes in Variability of the Business Cycle in the G7 Countries," Discussion Paper EI 2002-28, Econometric Insitute Report, Erasumus University.

Vogelsang, T. (1999): "Sources of nonmonotonic power when testing for a shift in mean of a dynamic time seris," *Journal of Econometrics*, 88, 283–299.

——— (2001): "Testing for a shift in trend when serial correlation is of unknown form," *unpublished manuscript.*

Xu, K.-L. (2008): "Bootstrapping autoregression under non-stationary volatility aaasssx," *Econometrics Journal*, 11, 1–26.

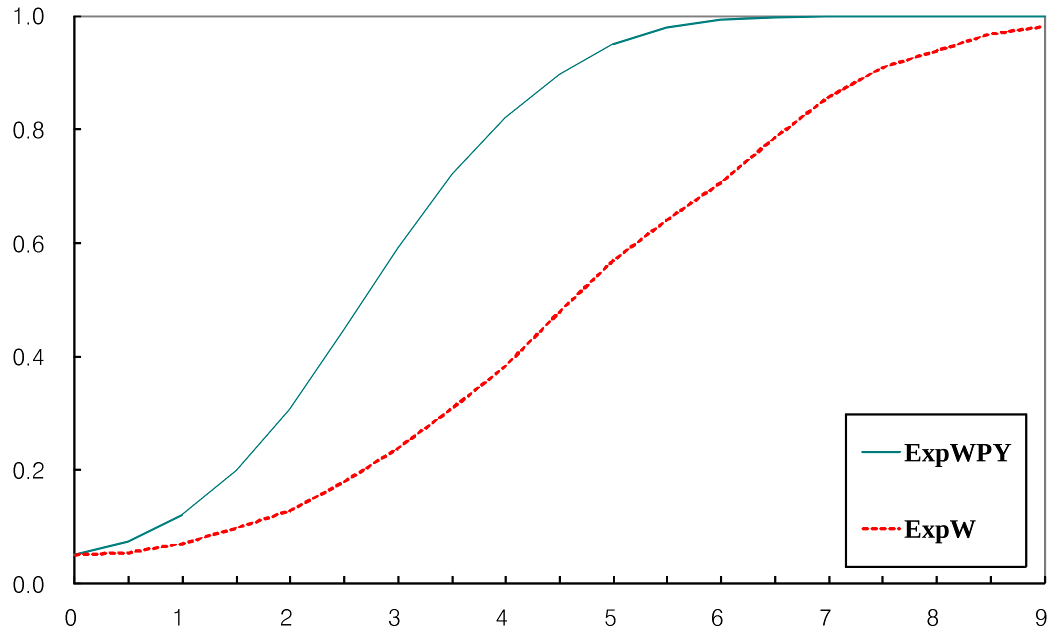


FIGURE 1. **Asymptotic Power Comparison between Perron and Yabu (2009)(ExpWpy) and the Bootstrap(ExpW)**

Table 1. **Monte Carlo Estimates of the Empirical Sizes (Linear Equation, T=100**)

(a) x_t **mean single break,** σ_t^2 **single break**

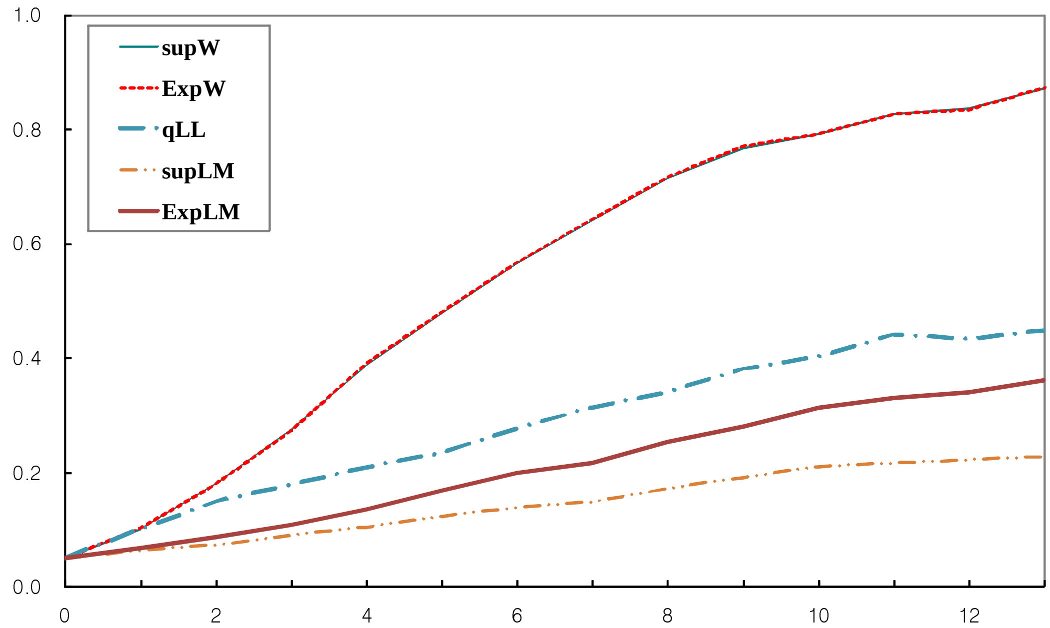
	supW	supLM	expW	expLM	qLL
classic	86.0	14.2	88.4	18.9	13.4
Hansen	46.0	12.2	45.6	16.8	17.8
SW	14.2	6.0	15.0	6.6	5.5

(b) **Monte carlo estimates of the empirical sizes (**x_t **variance single break,** σ_t^2 **single break**

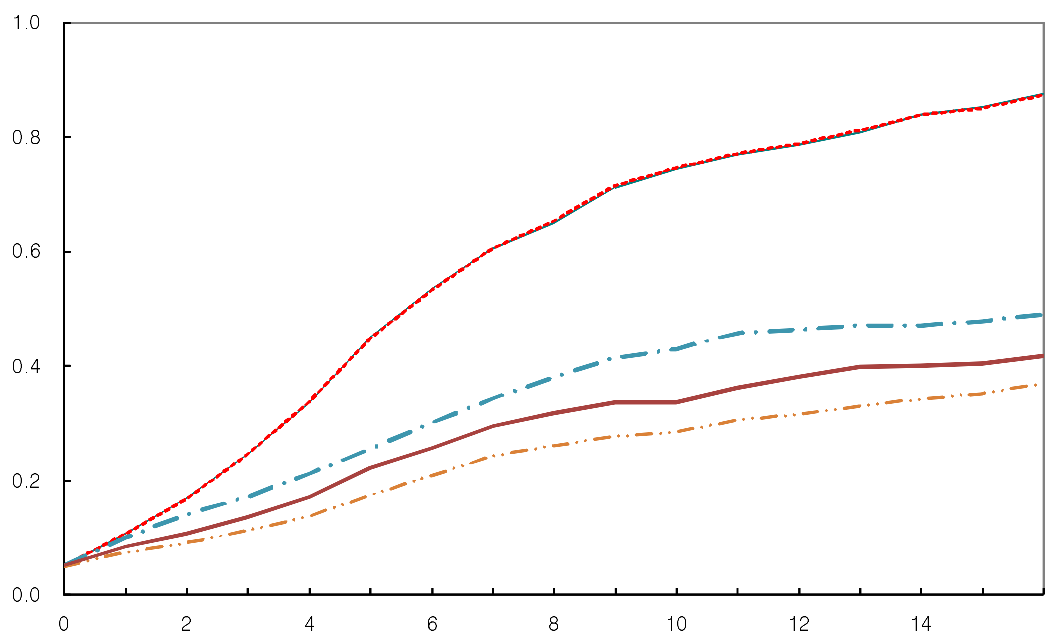
	supW	supLM	expW	expLM	qLL
classic	83.4	16.7	86.9	19.5	10.1
Hansen	41.2	13.2	45.6	18.5	14.8
SW	13.5	6.4	14.7	6.4	6.5

(c) **Monte carlo estimates of the empirical sizes (**σ_t^2 **random walk**

	supW	supLM	expW	expLM	qLL
classic	80.5	16.4	81.0	27.4	10.9
Hansen	36.5	17.5	42.0	16.6	15.2
SW	14.5	8.0	14.0	7.6	6.2

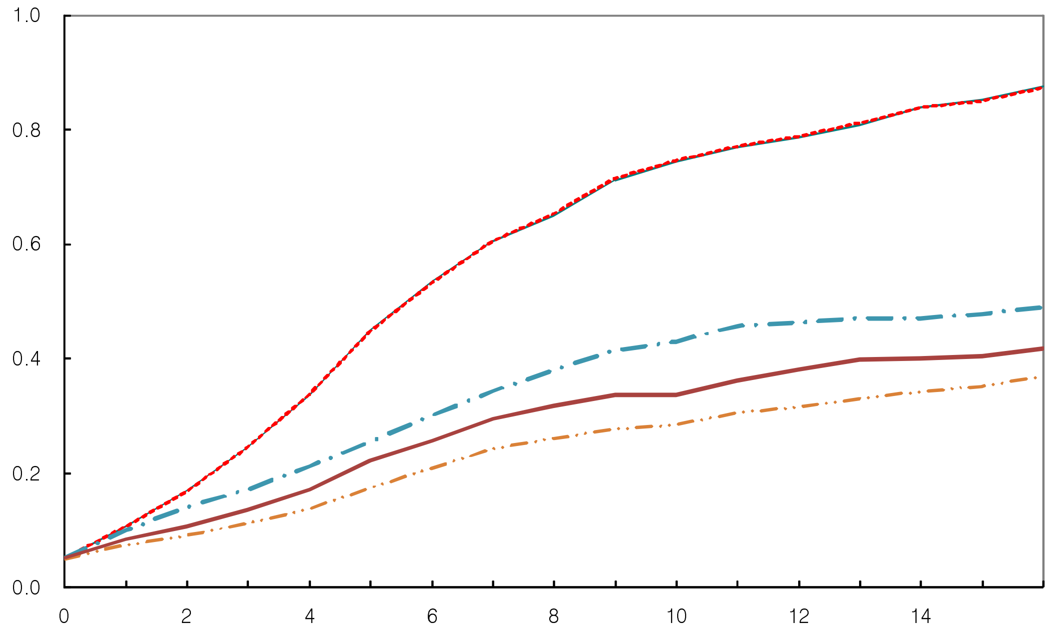


(a) Case I



(b) Case II

FIGURE 2. **Small Sample Powers, Single Break at random occurring time (1), T=100**

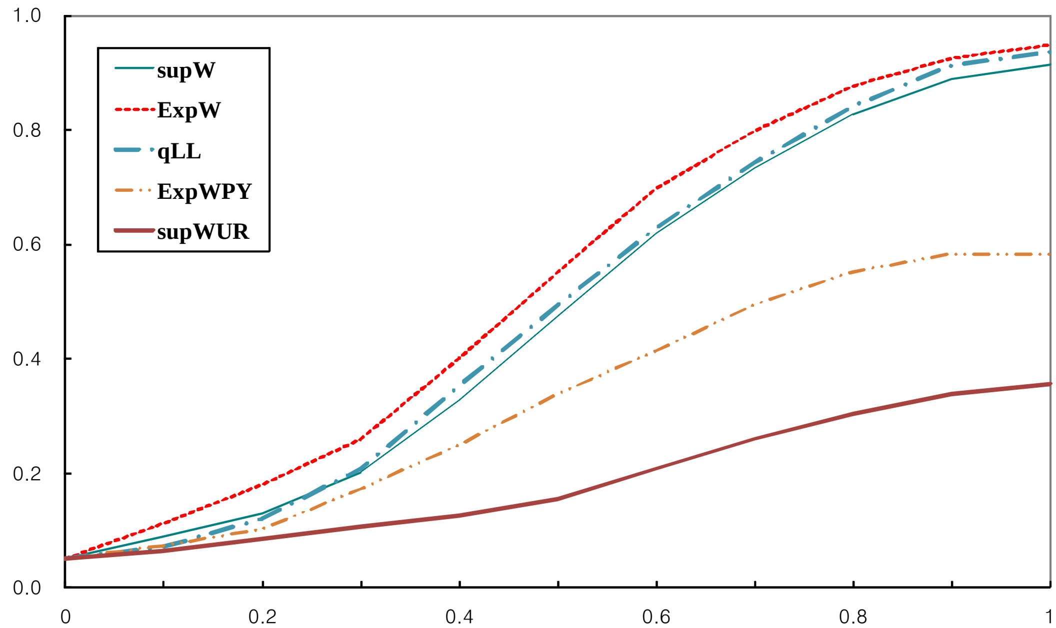


(a) Case III

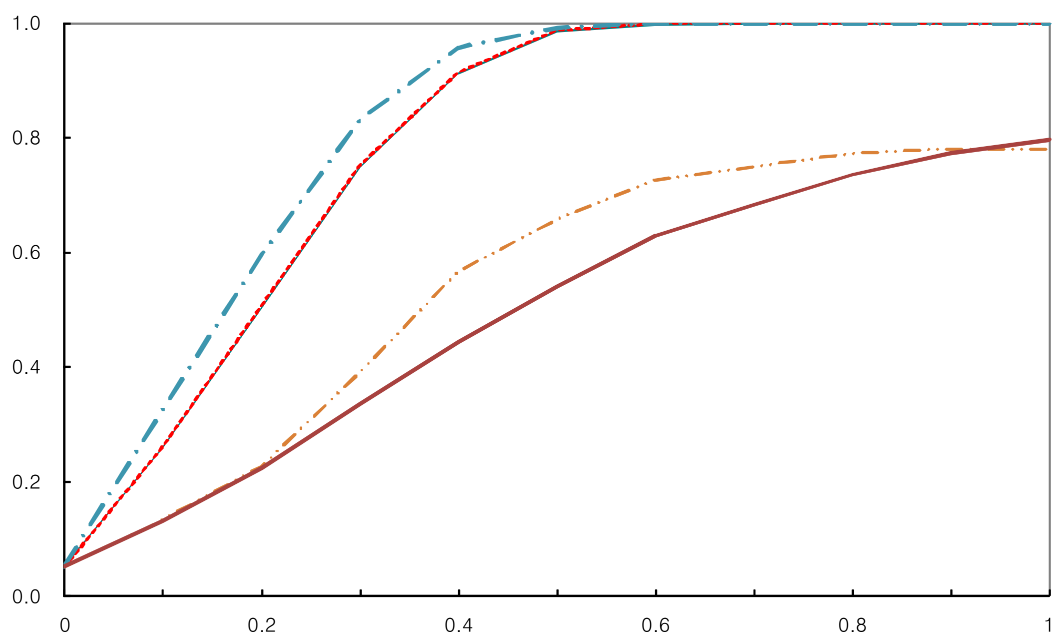
Figure 3. **Small Sample Powers, Single Break at random occurring time (2), T=100**

Table 2. **Monte carlo estimates of the empirical sizes(Linear Trend, T=200)**

	supW	supLM	expW	expLM	qLL	$expW_{PY}$	$supW_{UR}$
$\rho = 1$	5.0	5.5	6.75	5.5	4.5	89.5	3.5
$\rho = 0.95$	5.2	5.8	9.0	6.4	5.2	83.5	1.8
$\rho = 0.8$	4.8	5.6	6.6	5.2	4.5	49.8	0.6

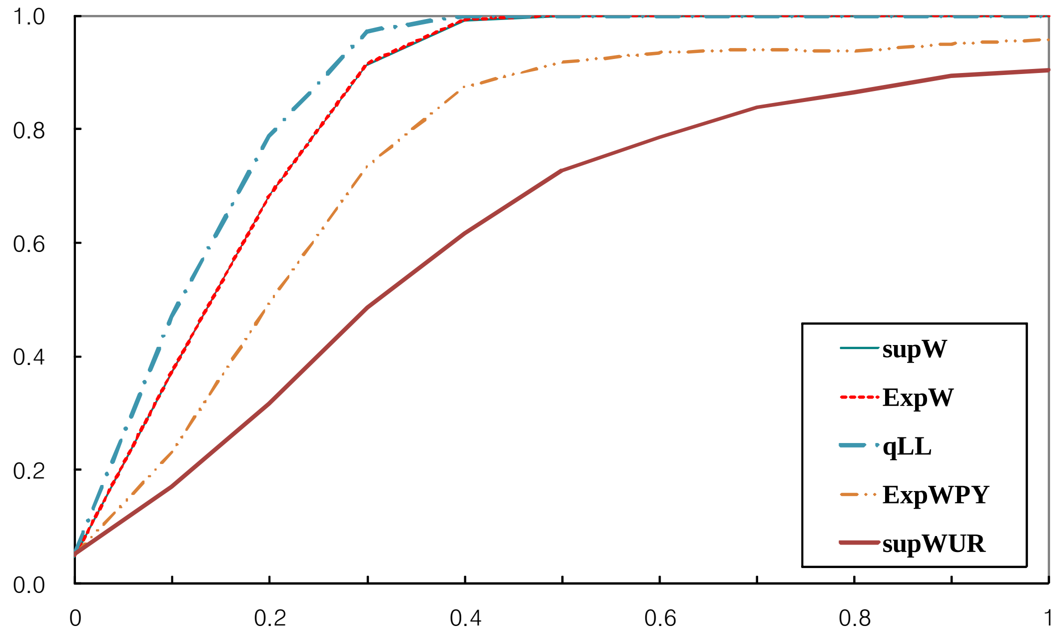


(a) $\rho = 1.0$

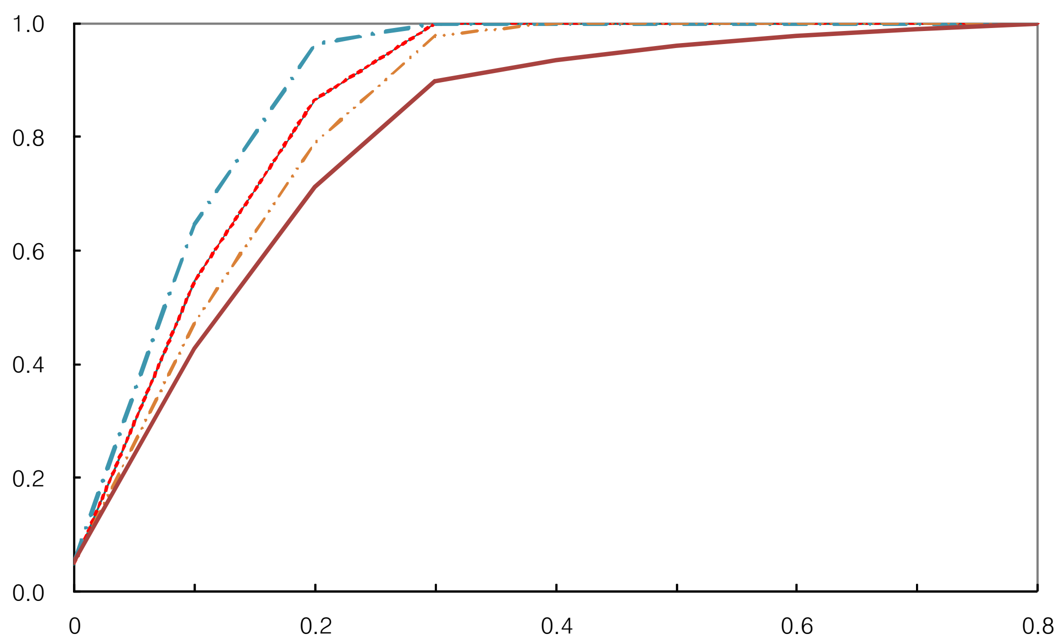


(b) $\rho = 0.95$

FIGURE 4. **Small Sample Powers, Linear trend (1), T=200**



(a) $\rho = 0.9$



(b) $\rho = 0.8$

FIGURE 5. **Small Sample Powers, Linear trend (2), T=200**