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Lagos-Wright Economy**

Parag Waknis
University of Connecticut
University of Massachusetts Dartmouth

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341 Mansfield Road, Unit 1063
Storrs, CT 06269-1063
Phone: (860) 486-3022
Fax: (860) 486-4463
<http://www.econ.uconn.edu/>

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Endogenous Monetary Policy: A Leviathan Central Bank in a Lagos-Wright Economy

Parag Wakis*^{*}

University of Connecticut

University of Massachusetts Dartmouth

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Abstract

This paper studies the nature of optimal monetary policy under a Leviathan monetary authority in a microfounded model of money based on Lagos and Wright (2005). Such a monetary authority is a reality whenever and wherever fiscal policy is a primary driver of the monetary policy. Under no commitment, we characterize and solve for a Markov perfect equilibrium as well as for equilibrium with reputation concerns. For the Markov equilibrium, a generalized Euler equation is derived to characterize optimal policy that trades off the current benefit of increasing consumption against the reduced ability to do so in the future. Under reputation equilibrium, centralized market interaction is modeled as an infinitely repeated game of perfect monitoring, between a Leviathan monetary authority (a large player) and the economic agents (small players). Such a game has multiple equilibriums but the large-small player dynamics pins down the equilibrium set of payoffs and features less than maximum inflation tax. Depending on how we interpret the Leviathan central bank, the factors determining the realized equilibrium differ. Higher fiscal profligacy of the underlying political authority leads to a higher monetary growth rate and inflation tax, while existence of threat of competition in case of a private money supplier or threat of external aggression in case of a self interested sovereign leads to a lower one. The realized equilibrium monetary growth rate and the associated inflation tax is thus, affected by the intensity of context contingent factors. Concentrating only on Markov strategies in this repeated game shows that the Markov perfect equilibrium features maximum inflation tax.

JEL Codes: E52, E61.

Keywords: Endogenous monetary policy, Leviathan, central bank, inflation tax, money search, Leviathan.

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1. Introduction

This paper studies the nature of optimal monetary policy under a utility maximizing monetary authority in a microfounded model of money. We call such a monetary authority as a Leviathan central bank to emphasize the use of new money to finance government spending. The model here is a version of Lagos and Wright (2005) with a utility maximizing central bank as an additional player in the centralized market. Analysis of monetary policy in such a setting is of interest from both macroeconomic and economic history perspective. From a macroeconomic point of view, a Leviathan central bank represents all those central banks around the world that finance a part or all of the government expenditure on goods and services (e.g. wages and salaries, defense goods) by printing new money or through seigniorage. This might happen as a result of unusual situations like the current financial crisis or because of lack of sufficient tax revenues, for example in the case of many developing countries.

From a historical perspective, a Leviathan monetary authority could proxy for a self interested sovereign. Such a representation could be valuable, especially given the recent interest in the effects of historical policies on today's economic outcomes.¹ In addition to these interpretations, a Leviathan central bank could also be thought of as a monopoly private money supplier.

Thomas Hobbes used the term 'Leviathan' to mean a strong and overarching central government or a sovereign. He was writing in the context of the English civil war and the associated turbulence in social and economic life. A strong centralized government was to facilitate order from such chaos. Following Oates (1985), the initial use of the term in public economics seems to be due to the writings of James Buchanan. He develops a theory of fiscal constitution to constrain the Leviathan's access to tax revenues or other fiscal instruments.

Among many such tax instruments that a Leviathan could possibly use (or abuse) is the inflation tax. Such a tax is a result of a government resorting to seigniorage to pay for its expenses. It seems to be important in economies that lack an alternative significant tax base. Many developing countries can be thought of as examples of such economies. For instance, in India the proportion of net central bank credit to the government to the reserve money, on an average for the period 1960-2002, was 81%.² The recent case of dollarization in Zimbabwe would be another example, although an extreme one. The Federal Reserve's recent policy of using its balance sheet as a tool of monetary policy could also serve as a good example of such a setting.³

¹It has been argued that limits on a self interested sovereign's ability to tax people may have had an impact on economic outcomes like capital accumulation and growth explaining the great divergence between Europe and the rest of the world (See Karayalçın (2008)). For a related theoretical investigation see Chatterjee and Lahiri (2010).

²Calculations based on the data in Handbook of Monetary Statistics of India, *Reserve Bank of India Annual Publications* available online.

³This is not the way the Federal Reserve has conducted its policy in past. To differentiate it from the prior practice, Curdia and Woodford (2010) characterize the current policy of the Federal Reserve as being unconventional.

How should one capture this fiscal-monetary policy nexus in a microfounded model of money? The construct of a utility maximizing central bank in this paper provides an answer. This central bank needs to consume in order to survive and uses its power to print money to do so. It represents the operational counterpart of a Leviathan government financing expenditure through seigniorage and hence the name, ‘Leviathan central bank’.

Irrespective of the interpretations, a utility maximizing monetary authority does face an intertemporal tradeoff that defines its decision problem. Printing more money today allows higher consumption today but only at the cost of reduced ability to raise it further in future. We use this tradeoff to characterize the optimal policy, assuming that the authority cannot commit itself to a future policy.

This intertemporal tradeoff allows the central bank’s decision problem to be modeled as a dynamic game with its successive self. We characterize and solve for a Markov perfect equilibria for this game assuming a differentiable policy function. Klein et al. (2008), Martin (2009) and Martin (2010) are examples that characterize policy in a similar way. However, they primarily talk about fiscal policy and the government is benevolent in their model. In this paper we talk about monetary policy and the central bank is not benevolent.

The characterization of policy involves derivation of what is called a **generalized Euler equation** from the first order conditions of the central banker’s decision problem. It shows how the central banker trades off the current benefits against the future losses to decide how much money to print today. Because use of money allows agents to optimize on trade in a decentralized settings, the equilibrium condition from their decision problem constrains the equilibrium path of the money supply that the central banker chooses. The future losses from choosing a higher monetary growth rate today emanate primarily from the fact that a depreciated money stock makes it difficult for the agents to smooth consumption in successive periods. It increases the work hours in the centralized market and reduces the quantity traded in the decentralized market. An increase in a monetary growth rate hence acts as an inflation tax on the agents. Today’s choice affects the future policy and agents expectations about future policy and therefore their current choices. So agents in the decentralized market trade lower or higher quantities depending on what will be the value of money in the second subperiod. I derive and prove the existence of a steady state policy function.

Though the derivations in this paper point at the nature of the Markov equilibrium, further characterization of such a policy function requires numerical methods. Alternatively, in this paper, we modify the decision horizon of the central bank in its interaction with the agents in centralized market and then use game theory to characterize the Markov perfect equilibrium. This equilibrium features the maximal inflation tax.

Notwithstanding the inability to commit to a policy, the central bank still could be con-

cerned with its long run reputation. In order to figure out the optimal policy under such a consideration, we look at the centralized market interaction as an infinitely repeated game of perfect monitoring between the Leviathan central bank and the continuum of economic agents. While doing so, we do have to think carefully about the nature of these two sets players in such a game. The economic agents behave competitively in the original LW setup as they do here. But the central bank cannot be expected to do so. Its choice of consumption is going to affect the price of the good it wishes to consume. Therefore, we model the central bank as a large or long run player and the agents as small or short run players.

A long lived player plays the game every period and maximizes the average discounted sum of payoffs in contrast to a short lived player who is concerned with only current period payoff (Mailath and Samuelson, 2006, p.61). The eventual stage game has a unique Nash equilibrium where the central banker chooses the monetary growth rate corresponding to maximum inflation tax on the agents and agents internalize the inflation tax by working more for every increase in the monetary growth rate, ensuring consumption smoothing. In the infinite repetition of such game, however, there are multiple equilibria. But the none of the monetary growth rates consistent with the equilibrium set of payoffs impose the maximum inflation tax featured in the stage game Nash equilibrium.

The Leviathan portrayal of a central bank in a Lagos-Wright economy might be unique to this paper but this is certainly not the first attempt to endogenize money supply in a search model. Berentsen (2006) does so for the model in Shi (1997). Araujo and Camargo (2008) endogenize money supply by allowing a monopoly of money issue to a long lived self interested agent. Their set up is closer in spirit to Kiyotaki and Wright (1991).

The remainder of the paper is organized as follows. Section 2 lays out the basic set up and describes the agent's decision problem. Section 3 describes the central bank's problem and characterizes the full commitment as well as the no commitment equilibrium. Section 4 concludes.

2. Baseline Model

The model here builds on the original Lagos and Wright (2005) (LW henceforth) setup. Goods and money are perfectly divisible. There are two subperiods, a day subperiod where special goods are traded in a decentralized market and a night subperiod where a general good is traded in a centralized Walrasian market. The decentralized market is characterized with trading frictions and hence money gets valued for the liquidity services it provides. The night trading, though centralized, is anonymous and is used by agents to trade in the general good and rebalance their portfolios. The economy is characterized by imperfect memory and record keeping

to rule out credit transactions as stressed by Kocherlakota (1998) and Wallace (2001).

2.1. Agent Behavior

Preferences for agents are given by $\mathcal{U}(x, h, X, H)$, where x and h (X and H) are consumption and labor hours during the day (night). They satisfy the usual continuity and differentiability conditions and there exists $X^* \in (0, \infty)$ such that $U'(X^*) = 1$ and $U'(X^*) > X^*$. There also exists $q^* \in (0, \infty)$ such that $u'(q^*) = c'(q^*)$.

$$\mathcal{U}(x, h, X, H) = u(x) - c(h) + U(X) - AH \quad (1)$$

To endogenize the money supply we add a central bank as a consuming agent to the basic LW economy. The day trade still remains decentralized and agents exchange special goods for money. The night trade will have the central bank as an additional agent that consumes via seigniorage or printing new money. Because the central bank needs to live, some growth in money supply is to be expected making the lower bound on the monetary growth rate greater than zero.

Agents have to decide how much of the general good to consume and how much money to take out of the market for the next day's use. They meet the central bank every alternate subperiod and have to make their decisions given the central bank's decision i.e. they take central bank's policy decision as given.

During the day, agents produce and consume highly specialized goods. For any two agents i and j , there are four possibilities. The probability that both consume what the other can produce (double coincidence of wants): δ . The probability that i consumes what j produces but not vice versa: σ . The probability that j consumes what i produces but not vice versa: σ . The probability that neither wants what the other produces: $1 - 2\sigma - \alpha\delta$. $u(q)$ and $c(q)$ is the utility from consuming and the cost of producing respectively. Let $V_t(m)$ be the value function for an agent carrying m money when he enters the decentralized market and $W_t(m)$ be the value function in the afternoon when he enters the centralized market.

$$\begin{aligned} V_t(m) = & \alpha\sigma \int \{u[q(m, \tilde{m})] + W_t[m - d(m, \tilde{m})]\} dF_t(\tilde{m}) \\ & + \alpha\sigma \int \{-c[q(\tilde{m}, m)] + W_t[m + d(\tilde{m}, m)]\} dF_t(\tilde{m}) \\ & + \alpha\delta \int B(m, \tilde{m}) dF_t(\tilde{m}) \\ & + (1 - 2\alpha\sigma - \alpha\delta)W_t(m) \end{aligned} \quad (2)$$

The four terms above are the expected payoffs to buying, selling, bartering ($B(m, \tilde{m})$) and not trading. m and $\tilde{m}$ are the buyer's and sellers money balances respectively. Agents enter the centralized market, observe the money growth rate, and solve the following:

$$W_t(m) = \max\{U(X) - H + \beta V_{t+1}(m')\} \quad (3)$$

$$\text{s.t. } X = H + \phi_t m - \phi_t m' \quad (4)$$

ϕ_t is the value of the money in the central market in terms of the general good. X is the consumption of general good and H are the hours worked. Hours worked are bounded above by some maximum allowable limit. m' is the money taken out of the centralized market.

2.1.1. Evolution of Money

The economy here starts with M_{t-1} instead of M_t as in the original model. The central bank participates in the period t centralized market by issuing new money making the total stock of money M_t . The additional new money is transferred to the agents in exchange of general good X . Because of quasilinear utility such consumption by the central bank has the same effect on money balances as a lumpsum transfer and therefore every agent carries out the same amount of cash out of the centralized market Lester et al. (2008). Thus, $t + 1$ decentralized trade starts with total stock of M_t (see Figure (1)).

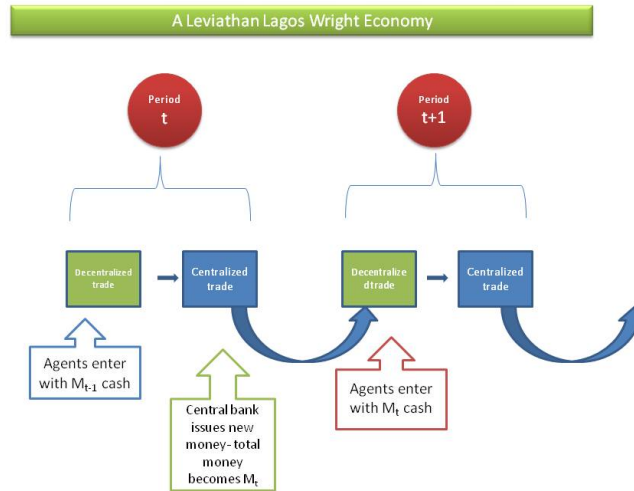


Figure 1: Model Structure

The terms of trade in the decentralized market are determined by a Nash bargaining process. Therefore, (q, d) maximizes:

$$[u(q) + W_t(m - d) - W_t(m)]^\theta [-c(q) + W_t(\tilde{m} - d) - W_t(\tilde{m})]^{1-\theta} \quad (5)$$

where θ is the buyer's bargaining power and the threat points are the continuation values. The terms of trade or money transferred in exchange for q is given by d . We follow the same solution procedure as Lagos and Wright (2005) to arrive at the equilibrium condition in one unknown. Agents take the policy of the central bank as given.

$$\phi_t = \beta \phi_{t+1} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} \right] \quad (6)$$

$$z(q_t) = \beta \frac{z(q_{t+1})}{(1 + \pi_t)} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} \right] \quad (7)$$

where $z(q)$ is given by:

$$z(q) = \frac{\theta c(q)u'(q) + (1 - \theta)u(q)c'(q)}{\theta u'(q) + (1 - \theta)c'(q)} = \phi M_{-1} \quad (8)$$

Gross inflation $(1 + \pi_t)$ is to come from the central bank's problem later and we assume that the agents take central bank's policy function as given. A monetary equilibrium is characterized as any path contained in $(0, q^*)$ and satisfying the above difference equation (7) in q . Here, q^* is the efficient quantity given by a double coincidence meeting. The central bank has to consider this difference equation while it chooses its consumption.

(6) tells us that the value of money today is equal to the discounted value of money tomorrow plus a liquidity factor, $\left[-\alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} \right]$, that captures the marginal benefit of holding the real balances in decentralized market (Nosal and Rocheteau (2010)). Williamson and Wright (2010) call this liquidity factor, *liquidity premium* $l(q_{t+1})$. We will use this terminology in favor of compactness of notation.

3. Central Bank's Problem

As mentioned before, in this model central bank is not benevolent. We consider two cases. In the first case we assume that the central bank can commit to a particular policy and hence chooses the policy and therefore sequences of money supply for the life time once for all. In the second case the central banker does not have the ability to commit across periods and hence chooses its policy every period. We characterize a Markov perfect equilibrium (MPE) as well as a repeated game equilibria for this case. The MPE concept of equilibrium is a refinement

designed to rule out any reputational mechanism that can sustain good equilibria (Klein et al. (2008)). The MPE equilibrium concept itself is due to Maskin and Tirole (2001). The repeated game structure is due to Fudenberg et al. (1990) and Mailath and Samuelson (2006).

3.1. Full Commitment

Under full commitment, the central banker chooses sequences of money supply for the life time once for all. He does consider (7) while doing so. We are looking for a decision rule of the form $M_{t+1} = rM_t$ as the central bank maximizes consumption in the centralized market.

Consumption of the central bank c_t^b as follows:

$$\begin{aligned}
 c_t^b &= (M_t - M_{t-1})\phi_t \\
 &= (M_{t-1}(r - 1) - M_{t-1})\phi_t \\
 &= (r - 1)M_{t-1}\phi_t \\
 &= (r - 1)r^{t-1}M_0\phi_t(r)
 \end{aligned} \tag{9}$$

Given the above expression for c_t^b , the central bank's optimization problem can be written as:

$$\mathcal{L} = \sum_{t=0}^{\infty} \mu(c_t^b) - \lambda \sum_{t=0}^{\infty} \left\{ \frac{\beta}{r} z(q_{t+1}) [1 + l(q_{t+1})] - z(q_t) \right\} \tag{10}$$

where the central bank chooses r given M_0 to optimize while keeping to the constraint defined by the agent's equilibrium condition. Agents take the policy r as given.

The FOC for the above problem is as follows:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial \mu}{\partial r} \frac{\partial c_t^b}{\partial r} + \lambda \frac{\beta z(q_{t+1}) [1 + l(q_{t+1})]}{r^2} = 0 \tag{11}$$

Assuming that $\mu(c_t^b) = \ln(c_t^b)$, we can use the above first order condition to solve for r to get:

$$r = \left[\left\{ \frac{1}{1-r} - \frac{\partial \phi_t}{\partial r} \frac{1}{\phi_t} \right\} \frac{1}{\lambda \beta z(q_{t+1}) [1 + l(q_{t+1})]} \right]^{\frac{1}{2}} \tag{12}$$

In steady state, (12) becomes:

$$r = \left[\left\{ \frac{1}{1-r} - \frac{\partial \phi}{\partial r} \frac{1}{\phi} \right\} \frac{1}{\lambda \beta z(q) [1 + l(q)]} \right]^{\frac{1}{2}} \tag{13}$$

Clearly from (13), r is given implicitly by the above equation. It depends negatively on the steady state q weighted by the liquidity premium and the value of money. It is also a decreasing function of λ which gives the sensitivity of r to changes in the steady state q .

It can be easily seen using the arguments of backward induction that a full commitment policy would suffer from time inconsistency. The central bank would always find in its interest to rescind on its promised consumption paths in the favor of increased utility today.

3.2. No Commitment-Repeated Game Structure

The interaction between the central bank and economic agents in the centralized market is repeated infinitely every second sub period. Hence, we model choice of monetary growth rate under no commitment as an infinitely repeated game. In contrast with the Markov perfect equilibrium, an infinitely repeated game can sustain a reputational equilibrium.

The economic agents continue to behave competitively but the central bank does not. This differential treatment is required to analyze the impact of central banks choice of monetary growth rate on this economy. We model it as a large player and the economic agents as small players. The central bank is large because it anticipates how the future policy depends on the current policy. The small players are small because they take future policies as given. Thus, the central bank behaves as *Stackelberg leader* taking the decisions considering the best responses of the agents.

Fudenberg et al. (1990) provide an analysis of games where the short run players play the game only once and the long run players play the stage game infinitely. Because the short run players are unconcerned about the future they play their best response moves and hence the equilibrium outcomes lie on their best response functions. There have been number of extensions and refinements of such a game and Mailath and Samuelson (2006) serves as a point of reference for this literature and the one on repeated games for building reputations. This section uses the modeling apparatus and terminology provided by them.

After every decentralized trading period, the centralized market opens with one long run player- the central bank and a $[0, 1]$ continuum of economic agents (short run players)⁴. In the stage game the central bank and the agents choose work hours and monetary growth rate to maximize consumption of the generic good X . The strategy or decision rule for the players are deduced from their utility and value functions. The decision rule for the agents asks them to respond by increasing the hours worked to positive changes in monetary growth rate. Any such increase in work hours for the agents can be interpreted as an inflation tax. The decision rule for the central bank asks it to print more money to increase its consumption.

⁴I use long run and short run players interchangeably with large and small players. In both the cases the reference is to the time horizon over which optimization takes place.

Each player's payoff depends on his own actions, the action of the central bank and the average of the small players' (all economic agents) actions. All players maximize average discounted sum of payoffs. The economic agents, being small or short lived players, do not affect the distribution of their actions and hence are "anonymous" and optimize myopically.

Note that because agents choose their myopic best responses, the payoffs for the central bank in both the equilibriums are *Stackelberg payoffs*. If the agents do not play best responses then the payoff range could be different. Thus, introducing short lived players restricts the possible payoffs for the long-lived player ((Mailath and Samuelson, 2006, p.61-67)).

The histories include the plays of the long run player (central bank) and only the distribution of play produced by the small players (agents). For simplicity, we assume perfect monitoring, in the sense that at the end of each period the small economic agents can observe the actions of the central bank and the central bank can observe the aggregate of the economic agents actions. The preferences of the agents and the central bank are as given in Sections 2 and 3.1.

3.3. Case 1: Infinite Period of Operation

In this section we assume that the central bank remains in operation for an infinite period. Consider the following one shot or stage game.

Players: One central bank (long run player) and $[0, 1]$ continuum of anonymous agents (short run player).

Strategy: Agents increase the work hours if the monetary growth rate increases. Central bank prints more money to increase its consumption.

Actions: Central bank chooses monetary growth rate, $\pi \in [\pi^{min}, \pi^{max}]$ and the agents choose $H \in [H^{min}, H^{max}]$ to maximize utility in the centralized market. The action sets are discrete and set of upper bounds are $S(\pi)$ and $S(H)$. Least upper bounds in the respective sets π^{max} and H^{max} ⁵.

Payoffs: Agents: $U(H, \pi) = U(X^* - H - \pi_t \phi_t M_{t-1})$

Payoffs: Central bank: $\mu(M_{t-1}, \pi) = u(\pi_t \phi_t M_{t-1})$.

Preferences: Preferences are given by the payoff functions. They are decreasing in H and π for the agents and increasing in π for the central bank.

Proposition 1. (Equilibrium-Stage game under no commitment): (H^{max}, π^{max}) is the only Nash equilibrium of the stage game.

⁵It is critical that H^{max} be the least upper bound. If not, we would have a corner solution where agents are not able to optimize the balances to be carried forward to the next day. This means that the distributions of money holdings will not be degenerate at the end of centralized market. Please see the relevant discussion in Lagos and Wright (2005).

Proof of Proposition 1. The agents myopically optimize by playing their best response for the stage game. If we plot the locus of the actions of agents against the locus of actions for the central bank, then till the maximum hours bind they both lie on the 45° line (Please see Appendix 3 for the figure). Any monetary growth rate off the line combined with hours worked on the line or any hours worked off the line combined with monetary growth rate on the line are not included in the best response function of the agents or the central bank. Thus, the 45° line also traces all possible equilibriums. However, only one of them is a Nash equilibrium of the stage game.

To see this point, note that because the agents always play their best responses, we only have to see if the central bank has any incentive for a unilateral deviation. It is easy to see that given the best responses of the agents, for all the choices of monetary growth rates but π^{max} , central bank has an incentive to deviate to a higher monetary growth rate. At π^{max} , $H = H^{max}$ and hence (H^{max}, π^{max}) is the only Nash equilibrium of the stage game. **QED.**

What happens when the game is infinitely repeated? Note that the usual folk theorem for infinitely repeated games is not applicable in this case because the presence of short run players constrains the payoffs of the long run players. We have to modify the definitions of feasible payoffs and minmax values for extending the folk theorem to such games (Fudenberg et al. (1990)).

In order to state the proposition for the equilibrium in the infinitely repeated game we also need some notation and additional definitions. Let a_i denote the action of player i , A_i being the set of pure actions, the set of mixed action profiles given by $\Delta(A_i)$ with typical element α_i . Let B be the correspondence that maps any mixed action profile for the central bank to the corresponding set of static Nash equilibria for the short-lived players. Then for the central bank, the long lived player, the minmax payoff ($\underline{v}_i$) is defined as follows:

$$\underline{v}_i = \min_{\alpha \in B} \max_{a_i \in A_i} \mu_i(a_i, \alpha_{-i}) \quad (14)$$

Similarly, the least favorable payoff ($\bar{v}_i$) for the central bank is given by:

$$\bar{v}_i = \sup_{\alpha \in B} \min_{a_i \in A_i} \mu_i(a_i, \alpha_{-i}) \quad (15)$$

Proposition 2. (Equilibrium Payoff Set): *Minmax and least favorable payoffs for the central bank are given by $\underline{v}_i = \pi^{min} \phi^h M_{t-1}$ and $\bar{v}_i = \pi^{max} \phi^{inln} M_{t-1} - \epsilon$ for some $\epsilon > 0$.*

Proof of Proposition 2.

The minmax payoff for the central bank is the lower bound on the payoff that he can obtain in the presence of short run players. Given that agents are constrained to Nash responses, the

lowest they can push the central bank is π^{min} . Hence, $\underline{v}_i = \pi^{min} \phi^h M_{t-1}$.

The least favorable payoff for the central bank according to Fudenberg et al. (1990) can be understood as the most the central bank can expect to get from profile $\alpha \in B$, if it takes the worst action that is called by α_i . Given the locus of equilibrium, the best the central bank can for itself is choose π^{max} . However, note that when some players are short run players, the equivalence between pure strategy stage game payoffs and mixed strategy stage game payoffs does not hold as it does if all the players were long run players. Therefore, some of the extreme payoffs like the one implied by (H^{max}, π^{max}) , in the presence of short run players, can be produced only by mixed strategies played by the long run player (Mailath and Samuelson, 2006, p.92).

The highest consumption is possible only with a mixed strategy whose outcome does not feature within the limit points. This is because, for constructing $\bar{v}$, we are choosing the least favorable mixture for the central bank and then selecting the profile from the short run players' best responses which maximizes this payoff. This means the payoff in this game will be lower than what the central bank could achieve with some mixture. Thus, the highest the central bank can reach is bounded away from π^{max} . Hence, $\bar{v}_i = \pi^{max} \phi^{inln} M_{t-1} - \epsilon$ for some $\epsilon > 0$. **QED**

According to the Proposition 2.7.2 in Mailath and Samuelson (2006), the following should hold:

Theorem 1. *Every subgame perfect equilibrium payoff for player i is less than or equal to $\bar{v}_i$.*

This implies that $\bar{v}_i$ is the upper bound on the payoffs of the long run player, the central bank. Equipped with all the notation and definitions above we can now state the theorem for repeated games with long run and short run players as follows:

Theorem 2. *Suppose there is one long-lived player, and $\underline{v}_i < \bar{v}_i$. For every $v_1 \in [\underline{v}_i, \bar{v}_i]$, there exists $\underline{\delta}$ such that $\forall \delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium of the repeated game with value v_1 . (Mailath and Samuelson, 2006, p.92) (Fudenberg et al., 1990, p.563)*

Put simply, as we go from the stage game to the infinite repetition, the number of equilibriums increases considerably. There will always be a δ such that a given equilibrium payoff can be supported as a subgame perfect equilibrium payoff. Thus, each of the equilibriums represented by the 45° line, excluding the stage game Nash equilibrium, can be supported as a subgame perfect equilibrium for some δ . Note that the above theorem is similar to the folk theorem for infinitely repeated games but the set of equilibrium payoffs is certainly different. The presence of short run players (agents) constrains the lower bound on the long run player's (central bank) payoff.

Given the above definitions and characterizations of payoffs, we now apply Theorem 2 to the infinitely repeated interaction of agents and the central bank in this Leviathan Lagos-Wright

economy under no commitment.

Proposition 3. (Long run-Short run player theorem for the Leviathan LW Economy): *Suppose there is one long-lived player- the central bank, and $\pi^{\min} \phi^h M_{t-1} < \pi^{\max} \phi^{\ln} M_{t-1} - \epsilon$. Then, for any $v_1 \in [\pi^{\min} \phi^h M_{t-1}, \pi^{\max} \phi^{\ln} M_{t-1} - \epsilon]$ there exists a $\underline{\delta}$ such that $\forall \delta \in (\underline{\delta}, 1)$, there is a subgame perfect equilibrium of the repeated game with payoff $[U(X^* - H^{v_1} - v_1), u(v_1)]$ obtained every period.*

Thus, as mentioned before, all the equilibriums on the 45° line except the Nash equilibrium of the stage game, can be supported as subgame perfect equilibriums for some appropriate degree of patience on part of the central bank. The more patient the central bank (discount factor close to 1), the more likely it will print less money and hence $(H^{\min}, \pi^{\min})$ or some other closer equilibria would likely be repeated. On the contrary, if the central bank is rather impatient then the equilibriums closer to the Nash equilibrium of the stage game would most likely be repeated.

Given that a Leviathan central bank is the operational counterpart of a government dependent on printing money to finance its expenditure, the level of patience (δ) could be interpreted as the extent of the political authority's dependence on seigniorage to finance its expenditure on goods and services. If the political authority cannot raise the required revenue by other means but printing new money, the Leviathan central bank will exhibit impatience and choose a higher monetary growth rate. On the other hand if the political authority has a recourse to other means of finances (e.g. taxes) then the central bank will exhibit patience and choose a lower monetary growth rate.

If we take the Leviathan central bank as a private money supplier, then the realized equilibrium would depend on variety of factors. The likelihood of its monopoly position being sustained will however turn out to be relatively more decisive factor among others. If such a money supplier does not see any immediate threat of competition then he will exhibit relative impatience and choose a higher monetary growth than if such a threat existed and was credible. Similar interpretation would hold if we take the Leviathan central bank to be a self interested sovereign in historic times. If the subjects are captive and there is little threat of external aggression then the realized equilibrium would feature a relatively higher monetary growth rate than otherwise.

Notwithstanding the above interpretations, the equilibrium payoff set does have a common feature- the maximum inflation tax is definitely lower than in the stage game Nash equilibrium. This is because the central bank is not able to choose the mixture necessary to achieve the monetary growth rate featured in the stage game Nash equilibrium. This also means that the agents also do not work to the maximum extent. However, this is not the only aspect of the inflation

tax. Because of the presence of a utility maximizing central bank, Friedman rule is not a feasible monetary policy in this economy. Hence, the hold up problem will always be present in this model unlike in the original one where the Friedman rule along with the assumption of full bargaining power to the buyer mitigates it.⁶ On the other hand in this economy, every increase in monetary growth rate is bound to constrain the buyer's real balances in the decentralized market and adversely affect the quantity exchanged (q) in such trades (the intensive margin) Berentsen et al. (2002). The numerical example in the next section brings out this aspect of the inflation tax more clearly.

3.3.1. Numerical Exercise

In what follows we adopt the quantitative analysis specifications from Lagos and Wright (2005) to this economy to illustrate its working numerically. In a steady state, the expenditure on the goods remains the same and hence we can use the equilibrium condition of the agents (Equation (7)) to derive the expression for quantity exchanged (q) in a bilateral meeting. We are also assuming that the buyer has full bargaining power implying $\theta = 1$. This in turn means that $z(q) = c(q)$ and $z'(q) = c'(q)$. Using this in (7) and solving for steady state q , we get the following:

$$q^{ss} = \left[\frac{1}{1 + \frac{1-\beta+\pi^{ss}}{\alpha\sigma\beta}} \right]^{\frac{1}{\eta}} \quad (16)$$

Because $z(q) = \phi M_{-1}$ and $z(q) = q$, we have $\phi = \frac{q}{M_{-1}}$. From the central bank's problem we know that $c^b = \pi z(q)$, which under the assumptions and working above means that $c^b = \pi q$. In addition we assume that $u(q) = \frac{q^{1-\eta}}{1-\eta}$ and $U(X) = B \log(X)$. The market clearing condition in the centralized market requires that $X^* + c^b = H^*$.

Given the expressions above and assuming that $\eta = 0.27$, $B = 1.96$, and $M_{-1} = 20$, the following table gives the steady state values of q , ϕ , c^b and H for different monetary growth rates.

Table 1:

π^{ss}	-.04	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.9999	1
q^{ss}	1	0.7434	0.3875	0.2227	0.1376	0.0898	0.0613	0.0433	0.0315	0.0235	0.0180	0.0139	0.0139
ϕ^{ss}	0.05	0.0371	0.0193	0.0111	0.0068	0.0044	0.0030	0.0021	0.0015	0.0011	0.0009	0.0006	0.0006
c_{ss}^b	- 0.04	0	0.0387	0.0445	0.0412	0.0359	0.0306	0.0260	0.0221	0.0188	0.0162	0.0139	0.0139
H^{ss}	1.92	1.96	1.9987	2.0045	2.0012	1.9959	1.9906	1.9860	1.9821	1.9788	1.9762	1.9739	1.9739

⁶The notion of holdup arises out of the fact that the agent who decided to carry money from period to another is making an investment with cost equal to ϕ . He can recover the cost only if he can reap the full benefit from the carried cash. This happens only if he has a full bargaining power and the Friedman rule is implemented (Lagos and Wright (2005)).

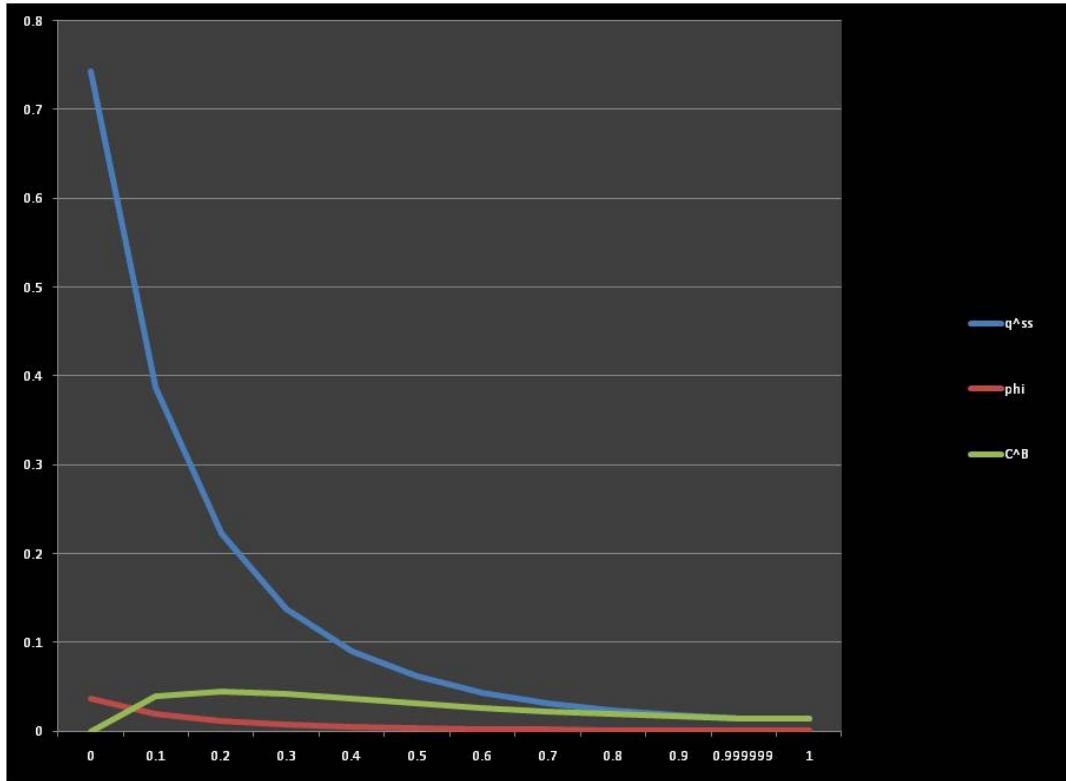


Figure 2: Effects of changing the monetary growth rate

In the benchmark Lagos and Wright (2005), $q^* = 1$ which is achieved only at the Friedman rule with buyer having the full bargaining power. From column 1 we can see that, in this model, for $q^* = 1$, the Leviathan central bank should contract the money supply at the rate of 4%.

Figure (2) is based on the numbers in Table 1 . It brings out the effects of changing the monetary growth rate in this economy more clearly. The shapes of all the three series conform to the theoretical shapes in the figure (4) . We can see that consumption of the central bank follows a hump shape and the value of money declines for successive increases in the steady state monetary growth rate. In addition to these relationships, the blue line shows the behavior of the quantity exchanged in the bilateral meetings. It decreases with the increase in the steady state central bank consumption. This is the adverse effect of a higher monetary growth rate on the intensive margin mentioned above. Thus, the inflation tax not only implies that the agents work more in the centralized market but they also have to suffer reduced consumption in the decentralized market.

A simple welfare comparison of the agents and the Leviathan central bank at various steady

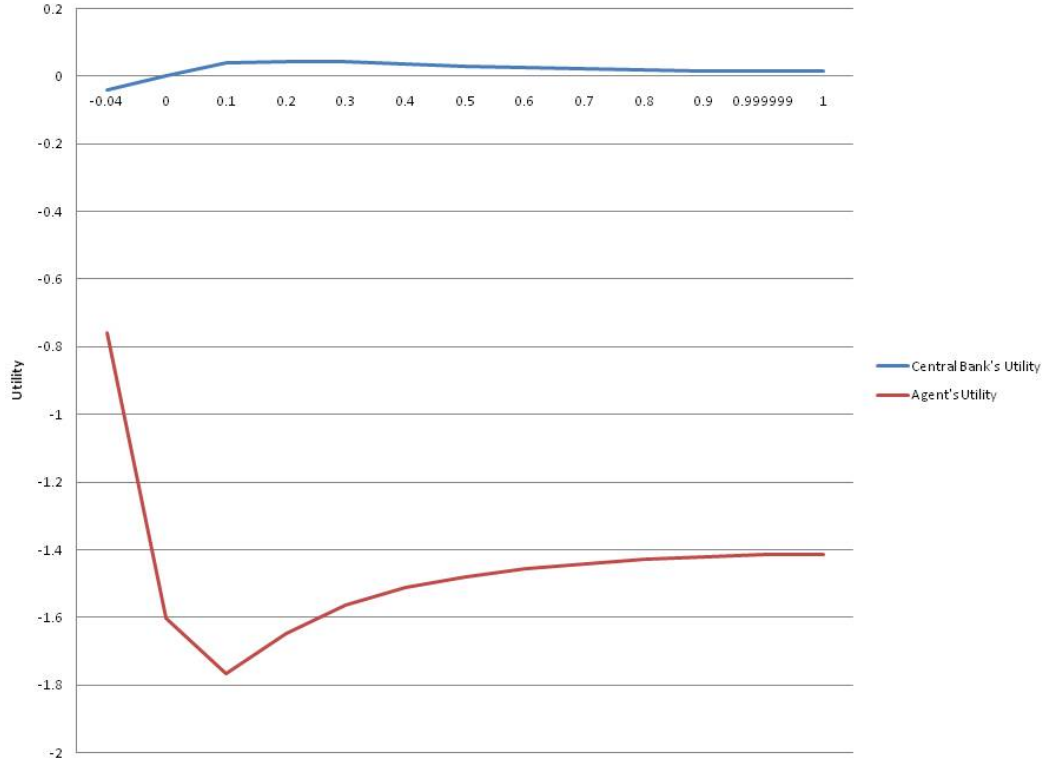


Figure 3: Welfare comparisons under various monetary growth rates

state monetary growth rates is given in Figure (3). It uses the functional forms for agent's utility function and parameter values mentioned above. Central bank's utility is assumed to be linear.

From the figure it is easy to see that when the central bank's utility is highest, a typical agent's utility is the lowest. It only starts to improve as the welfare of the central bank starts to deteriorate. Clearly when the central bank actually contracts the money supply, agents enjoy maximum welfare.

Different values for the initial parameters (η , B and M_{-1}) do not change the qualitative nature of these relationships. For example, a higher initial money supply is associated with a lower value of money to start with. But apart from such level differences, the curves look exactly the same as for the values assumed above.

3.4. Case 2: Finite Period of Operation

In this case we only modify the game of Case 1 by allowing only a fixed term of operation for a given central bank. What is the nature of equilibrium in such a situation? The stage

game remains the same and hence the corresponding Nash equilibrium also remains the same, (H^{max}, π^{max}) .

Because every central bank only lasts for a certain period, the game that describes the interaction between agents and the central bank becomes a finitely repeated game. This is despite the assumption that the agents are infinitely lived. To see what would be the equilibrium for the finitely repeated game, consider the following proposition for a finitely repeated game with a unique stage game equilibrium:

Theorem 3. *If a stage game G has a unique Nash equilibrium then, for any finite T , the repeated game $G(T)$ has a unique subgame perfect equilibrium outcome: the Nash equilibrium of G is played in every stage. (Gibbons, 1992, p.84)*

Given that the stage game has a unique Nash equilibrium, the application of the above proposition implies that (H^{max}, π^{max}) is the equilibrium of this finitely repeated game and is unique. The proof is simply applying the logic of backward induction. If there is no profitable deviation from the stage game Nash equilibrium that the central bank can find given the agents' best responses, then that would be the play in period T . This implies that the stage game Nash equilibrium will be played in period $T - 1, T - 2, \dots$, and so on till the first period. This makes (H^{max}, π^{max}) the unique Nash equilibrium of game between agents and the central bank, repeated for the period of the later's term of operation.

Thus, in case of finite period of operation the central bank chooses the maximum monetary growth rate thereby imposing the maximum inflation tax on the agents.

3.5. No Commitment-Markov Perfect Equilibrium

Under this equilibrium, the central bank takes decisions based only on the state variable. A Markov equilibrium reduces the information requirement to reach equilibrium decisions. As earlier, the bank chooses a monetary growth rate to maximize utility $\mu(c_t^b)$ from consumption in the centralized market. The value function for this problem can be written as follows:

$$C_t(M_{t-1}) = \max_{\pi_t} [\mu(c_t^b) + \beta C_{t+1}(M_t)] \quad (17)$$

where M_{t-1} is the state variable and π_t is the control variable.

The consumption of the central bank can be expressed as follows:

$$\begin{aligned}
c_t^b &= (M_t - M_{t-1})\phi_t \\
&= (M_{t-1}(1 + \pi_t) - M_{t-1})\phi_t \\
&= \pi_t z(q_t) \\
&= [G(M_{t-1}) - 1]z(q_t)
\end{aligned} \tag{18}$$

where $1 + \pi_t = G(M_{t-1})$ is the policy rule that the central bank solves for while optimizing. It is assumed that the policy function depends differentiably on the money stock. The agents take this policy rule as given and react accordingly. Considering this, equilibrium q is given by:

$$z(q_t) = \beta \frac{z(q_{t+1})}{(G(M_{t-1}))} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} \right] \tag{19}$$

The central bank solves the optimization problem knowing that (19) holds.

Definition 1. A Markov-perfect equilibrium is a set of functions $\{C, G\}$: such that for all M ,

$$G(M_{t-1}) = \arg \max_{\pi} \mu(c_t^b) + \beta C(M_t)$$

subject to

$$z(q_t) = \beta \frac{z(q_{t+1})}{(G(M_{t-1}))} \left[1 - \alpha\sigma + \alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} \right]$$

and

$$C(M_{t-1}) = \max_{\pi_t} [\mu(c_t^b) + \beta C(G(M_t))]$$

Using this information we can write the central bank's optimization problem as follows:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \mu(c_t^b) - \beta^{t+1} \lambda_{t+1} \left\{ z(q_t - \beta \frac{z(q_{t+1})}{(G(M_{t-1}))} [1 + l(q_{t+1})]) \right\} \tag{20}$$

Following Klein et al. (2008) and Martin (2009), we can express the constraint in (20) as $\eta_t(G(M), z(q), \phi) = 0$. Using (Chow, 1997, p.22), we have the following as FOC's for the central bank's problem:

$$\frac{\partial \mathcal{L}}{\partial G(M_{t-1})} = \frac{\partial \mu}{\partial G(M_{t-1})} \frac{\partial c_t^b}{\partial G(M_{t-1})} + \beta \lambda_{t+1} \eta_{t+1}^{G(M_{t-1})} = 0 \tag{21}$$

$$\lambda_t = \frac{\partial \mu}{\partial z(q_t)} \frac{\partial c_t^b}{\partial z(q_t)} + \beta \lambda_{t+1} \eta_{t+1}^{z(q_t)} \tag{22}$$

Adding the two FOC's above gives us what is called the **generalized Euler equation (GEE)** for the central bank⁷. It captures the tradeoff we are interested in.

$$\frac{\partial \mu}{\partial G(M_{t-1})} \frac{\partial c_t^b}{\partial G(M_{t-1})} = \left[\lambda_t - \beta \lambda_{t+1} \eta_{t+1}^{G(M_{t-1})} \right] - \left[\beta \lambda_{t+1} \eta_{t+1}^{z(q_t)} + \frac{\partial \mu}{\partial z(q_t)} \frac{\partial c_t^b}{\partial z(q_t)} \right] \quad (23)$$

(23) above captures the tradeoff from selecting a higher monetary growth rate today in the following sense. The terms in the first parentheses on the right capture the net effect of choosing a policy of higher consumption today. The first term is the marginal value of the relaxed constraint today and the second term is the discounted marginal value of tightened constraint tomorrow. A different policy that allows for higher consumption today means higher inflation next period and hence a tighter constraint. The increase in utility from choosing such policy today is therefore, the sum of these two effects adjusted for the change in net utility from equilibrium conditions of the agents (second parentheses on the right).

3.5.1. Steady State Analysis

Given that the central bank trades off the current benefit versus the future losses of increasing the monetary growth rate today, how does the optimal policy look like in steady state? To solve for steady state policy function we solve the system of equations comprised of (7) and the first order conditions given by (21) dropping the time subscripts from the relevant variables (Chow, 1997, p.23). We obtain the following expression for the steady state policy function under no commitment (See Appendix A for the working).

$$G(M_{-1}) = \left\{ (G(M_{-1}) - 1) [1 + l(q)] \frac{\beta^2}{\beta - 1} \right\}^{\frac{1}{2}} \quad (24)$$

It is clear from (24) depends on its own value, the steady state liquidity premium and a term based on β . It is defined as a fixed point satisfying the above equation. A higher liquidity premium suggests that the marginal value of any unspent money carried next period is higher relative to today and hence a higher monetary growth rate could be sustained. The policy function also displays the trade off between choosing a higher monetary growth today. A higher liquidity premium implies higher monetary growth rate but a higher monetary growth rate implies a lower liquidity premium in future. Optimal policy will be defined by this tradeoff.

Note that q is a function of M_{-1} and can be written as $q = h(M_{-1})$. Given this, (24) can be written as:

$$G(M_{-1}) = \left\{ (G(M_{-1}) - 1) [1 + l(h(M_{-1}))] \frac{\beta^2}{\beta - 1} \right\}^{\frac{1}{2}} \quad (25)$$

⁷It is called so due to the presence of the derivative of an equilibrium function. (Martin, 2009, p.8)

Now (25) becomes a function in one unknown and hence can be expressed as:

$$\mathcal{T}G(M_{-1}) = G(M_{-1}) \quad (26)$$

where, $\mathcal{T}$ is given by the right side of (25), implying that the function G is mapped into itself. It can be shown that $\mathcal{T}$ is a contraction mapping and hence by the contraction mapping theorem, there exists a fixed point that satisfies (26). (See Appendix for the proof.)

3.5.2. Markov perfect equilibrium using the repeated game

We can use the game described above with some modifications, to actually characterize the Markov perfect equilibrium. However, instead of solving for a policy function explicitly as in the previous section, we would have to simply assume that both agents and the bank take period by period decision. This means that the stage game between the central bank and the agents changes from period to period, at least for the central bank in terms of the payoffs. Hence, we can think of this game as a dynamic game. Typically in such games the current state and actions determine the current payoffs ruling out reputational dynamics.

We have to show that the decision rules or strategies specified for both these players are Markov strategies. The decision rule for the central bank is to increase the monetary growth rate to increase the consumption. The decision rule for the agents says that they have to work more if the monetary growth rate increases.

The strategy profile σ is a Markov strategy if for any two ex post histories, $\tilde{h}^t$ and $\tilde{h}^\tau$ of the same length and terminating in the same state, $\sigma(\tilde{h}^t) = \sigma(\tilde{h}^\tau)$. The strategy profile σ is a stationary Markov strategy if for any two ex post histories, $\tilde{h}^t$ and $\tilde{h}^\tau$ (of equal or different lengths) and terminating in the same state, $\sigma(\tilde{h}^t) = \sigma(\tilde{h}^\tau)$.

Proposition 4. (Markov and Stationary Markov Strategies): *The decision rule for the central bank in the stage game above is Markov as well as stationary Markov.*

Proof of Proposition 4. Let there be two ex post histories, $\tilde{h}^1$ and $\tilde{h}^2$ of either similar or different length. Let $\sigma =$ (increase the money growth rate to increase consumption) and $\sigma' =$ (do nothing). Also, let the states be ordered according to the consumption from lower to higher. Suppose that at the start of history $\tilde{h}^1$, consumption is c_1 and at the start of the history $\tilde{h}^2$, consumption is c_2 ending in a state with consumption c_3 . Note that because states are ordered, $c_1 < c_2 < c_3$.

It is easy to see that only if σ was followed in both the cases, the state c_3 could be reached. This is assuming that agents still have some room to adjust their work effort (inflation tax). If σ' was followed instead, then the state at the end of histories would be the respective states. This

holds for any two histories irrespective of their lengths being equal or not. Hence, σ is not only Markov but stationary Markov. **QED**

What would be the equilibrium of the repeated game if we only consider Markov strategies? In a repeated game with only Markov strategies, a Markov equilibrium must play a stage game Nash equilibrium and for a stationary Markov strategies, it should play the same one in every period (Mailath and Samuelson, 2006, p.178). Thus, given that the unique Nash equilibrium of the stage game in the previous section features seigniorage maximizing monetary growth rate, a stationary Markov perfect equilibrium will always feature the maximum inflation tax. Thus, (H^{max}, π^{max}) is a Markov perfect equilibrium of the repeated game.

3.6. Summary

Under different timing protocols, we characterized the optimal policy for a utility maximizing central bank. We demonstrated that the policy under full commitment suffers from time inconsistency and hence is not credible. Under no commitment we considered two cases: in one case reputation concerns were important and in the other they were not. To characterize these cases, we used a game theoretic approach. Under reputation, we modeled the central bank as a long lived player and the agents as a continuum of short lived players and used a theorem by Fudenberg et al. (1990) to show there are multiple equilibria possible.

The no commitment case without reputation was analyzed using the concept of a Markov perfect equilibrium. The generalized Euler equation approach helped us to show that a policy function under Markov perfect equilibrium exists, but any further specification of such a function required use of numerical methods. However, we were able to come up with an alternative analytical characterization of a Markov equilibrium by modifying the game between central bank and the agents. We showed that if both the players were only allowed period by period decisions, then a Markov perfect equilibrium featured highest possible consumption for the central bank and hence highest inflation tax on agents. Limiting the period of operation for the central bank delivered a similar result.

4. Conclusion

A Leviathan central bank is a reality wherever the governments rely heavily on seigniorage. This paper studies the money growth chosen by such a central bank in different situations. Under no commitment, we characterize a Markov perfect equilibrium as well as equilibria in a repeated games setting with long run and short run players. Under Markov equilibrium, the central bank decides on its policy by optimally trading off the current benefit of higher consumption against

the reduced ability to raise it further in future. A higher monetary growth rate today depreciates the value of money stock in future making it difficult for the agents as well as the central bank to smooth consumption in successive periods. Hence, a steady state monetary growth rate (through the policy function) is determined by the marginal value of any unspent money carried over to the next period and the degree of patience of agents.

Under repeated game setting, the policy of the central bank is constrained by the nature of responses of the economic agents and its ability to randomize over the relevant range of the monetary growth rates. If the central bank lived infinitely then its optimal choice corresponds to a lower consumption than the unique stage game Nash equilibrium and hence a lower inflation tax on the agents. The actual equilibrium would depend on the kind of incentives the central bank faces and would change according to what we take the Leviathan central bank to represent. The extent of fiscal profligacy of the underlying political authority in today's world, existence of threat of competition in case of a private money supplier or threat of external aggression in case of a self interested sovereign are some of the factors that will determine the realized equilibrium under different interpretations. Irrespective of these interpretations, if the period of operation is finite or if we consider only Markov strategies then the stage game unique Nash equilibrium is also the equilibrium under finite repetition and a Markov perfect equilibrium. The central bank gets to consume at the maximum possible level and agents bear the maximum inflation tax.

Thus, under a Leviathan setting, optimal monetary policy is characterized by multiple equilibriums under reputation and an unique one with Markov strategies. The actual realized equilibrium under reputation, however, would depend on context contingent factors and period of operation of the central bank. Any further narrowing down of possibilities will require some modifications to the model or allowance for some type of equilibrium refinement mechanism.

A

A1. Full Commitment Policy Function

Given the definition of c_t^b under full commitment we have the following:

$$\frac{\partial c_t^b}{\partial r} = \phi_t M_0 [t(r)^{t-1} - (t-1)r^{t-2}] + (r-1)^{t-1} M_0 \frac{\partial \phi_t}{\partial r} \quad (27)$$

Now manipulating (11), we get:

$$\frac{\partial \mu}{\partial r} \frac{\partial c_t^b}{\partial r} = -\lambda \frac{\beta z(q_{t+1}) [1 + l(q_{t+1})]}{r^2} \quad (28)$$

Assuming $\mu(c_t^b) = \ln(c_t^b)$ and using (27) we have the following:

$$\frac{\phi_t M_0 [t(r)^{t-1} - (t-1)r^{t-2}] + (r-1)^{t-1} M_0 \frac{\partial \phi_t}{\partial r}}{(r-1)r^{t-1} M_0 \phi_t(r)} = -\lambda \frac{\beta z(q_{t+1}) [1 + l(q_{t+1})]}{r^2} \quad (29)$$

$$\Rightarrow -\frac{1}{r-1} + \frac{\partial \phi_t}{\partial r} \frac{1}{\phi_t} = -\lambda \frac{\beta z(q_{t+1}) [1 + l(q_{t+1})]}{r^2}$$

$$\Rightarrow -\frac{1}{r-1} + \frac{\partial \phi_t}{\partial r} \frac{1}{\phi_t} = -\lambda \frac{\beta z(q_{t+1}) [1 + l(q_{t+1})]}{r^2}$$

$$\Rightarrow r = \left[\left\{ \frac{1}{1-r} - \frac{\partial \phi_t}{\partial r} \frac{1}{\phi_t} \right\} \frac{1}{\lambda \beta z(q_{t+1}) [1 + l(q_{t+1})]} \right]^{\frac{1}{2}} \quad (30)$$

In steady state, (30) becomes:

$$\Rightarrow r = \left[\left\{ \frac{1}{1-r} - \frac{\partial \phi_t}{\partial r} \frac{1}{\phi} \right\} \frac{1}{\lambda \beta z(q) [1 + l(q)]} \right]^{\frac{1}{2}} \quad (31)$$

A2. Steady State under No Commitment MPE

After dropping the relevant time subscripts (21) becomes:

$$\frac{\partial \mu}{\partial G(M_{-1})} \frac{\partial c^b}{\partial G(M_{-1})} + \beta \lambda \eta^{G(M_{-1})} = 0 \quad (32)$$

$$\frac{\partial \mu}{\partial z(q)} \frac{\partial c^b}{\partial z(q)} + \beta \lambda \eta^{z(q)} = \lambda \quad (33)$$

The first equation above implies:

$$\lambda = - \frac{\partial \mu}{\partial G(M_{-1})} \frac{\partial c^b}{\partial G(M_{-1})} \frac{1}{\beta \eta^{G(M_{-1})}} \quad (34)$$

Using the above equation in the second FOC and after some manipulation above we get:

$$\frac{\frac{\partial \mu}{\partial z(q)}}{\frac{\partial \mu}{\partial G(M_{-1})}} \frac{\frac{\partial c^b}{\partial z(q)}}{\frac{\partial c^b}{\partial G(M_{-1})}} = \frac{\eta^{z(q)}}{\eta^{G(M_{-1})}} - \frac{1}{\beta \eta^{G(M_{-1})}} \quad (35)$$

From the definition of c^b we know that $\frac{\partial c^b}{\partial G(M_{-1})} = G(M_{-1})' z(q)$ and $\frac{\partial c^b}{\partial z(q)} = [G(M_{-1}) - 1]$. Also given the definition of η we have the following:

$$\begin{aligned} \eta^{G(M_{-1})} &= z(q)[1 + l(q)]\beta \frac{G'(M_{-1})}{G(M_{-1})^2} \\ \eta^{z(q)} &= -1 \end{aligned}$$

Using this information in (35) above we get:

$$\frac{\frac{\partial \mu}{\partial z(q)}}{\frac{\partial \mu}{\partial G(M_{-1})}} \frac{\frac{\partial c^b}{\partial z(q)}}{\frac{\partial c^b}{\partial G(M_{-1})}} = \left\{ \frac{\beta G(M_{-1})^2}{G'(M_{-1})\beta^2 z(q)[1 + l(q)]} - \frac{G(M_{-1})^2}{G'(M_{-1})\beta^2 z(q)[1 + l(q)]} \right\} \quad (36)$$

Solving the above equation for $G(M_{-1})$, we get:

$$G(M_{-1}) = \left\{ (G(M_{-1}) - 1)[1 + l(q)] \frac{\beta^2}{\beta - 1} \right\}^{\frac{1}{2}} \quad (37)$$

A2.1. Proof that $\mathcal{T}$ is a contraction.

Proposition 5. (Unique Policy Function): $\mathcal{T}$ is a contraction

Proof of Proposition 5.

If $\mathcal{T}$ satisfies the Blackwell sufficiency conditions, then it is a contraction and the contraction mapping theorem applies. An operator has to satisfy the conditions of monotonicity and discounting.

Because of concavity for any $G_2 > G_1$, $\mathcal{T}$ implies $\mathcal{T}G_2 > \mathcal{T}G_1$ satisfying the monotonicity condition.

Because $\mathcal{T}$ is a concave function with power less than 1 and that the value of the terms inside the bracket are always going to less than $G(M_{-1})$, it should be true that:

$$\mathcal{T}(G(M_{-1}) + a) \leq \mathcal{T}(G(M_{-1})) + a$$

where $a \geq 0$. Given this we can definitely find a $\zeta \in (0, 1)$ such that:

$$\mathcal{T}(G(M_{-1}) + a) = \mathcal{T}(G(M_{-1})) + \zeta a$$

Thus, $\mathcal{T}$ is a contraction. Given that this contraction is defined on variables belonging to compact sets, according to the Banach fixed point theorem, we can show that there is a unique policy function satisfying (25) (Corbae et al., 2009, p.122, 282). **QED**

A3. Application of Theorem for Long run and Short run players

A3.1. Dynamics of H , π and ϕ under no commitment

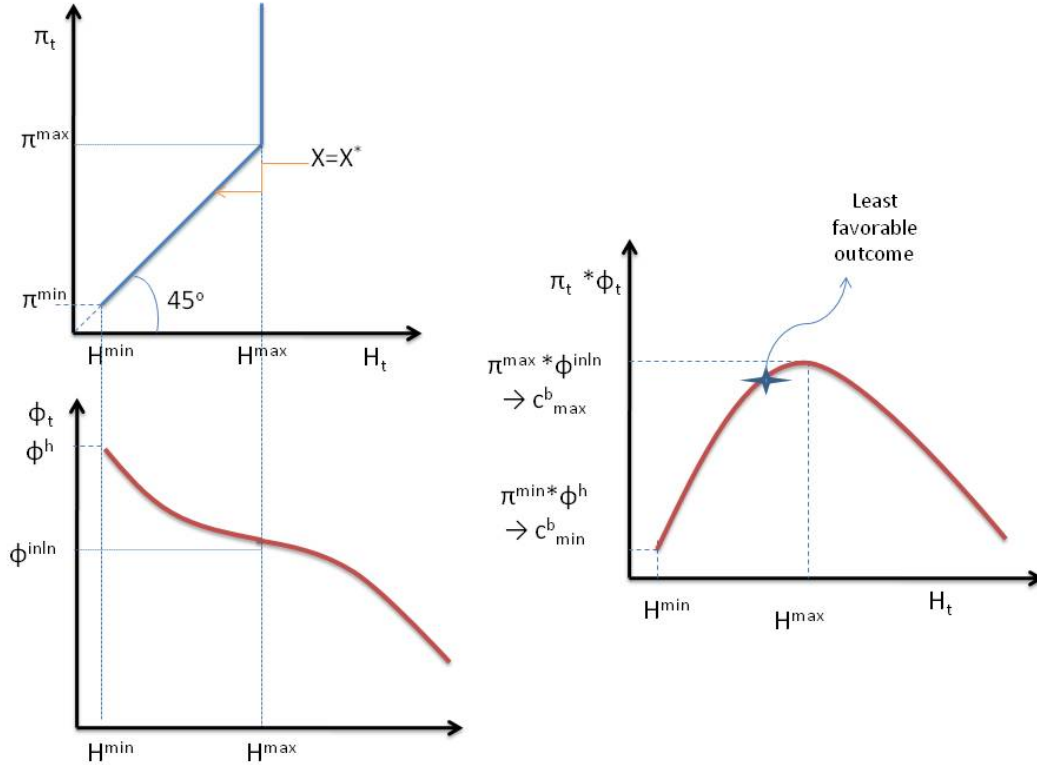


Figure 4: Central bank's consumption under no commitment

When the central bank increases the monetary growth rate in order to increase its consumption, agents work more to finance this consumption. So the inflation rate rises but probably at a slower rate than the monetary growth rate. However, after the maximum hour limit is hit, it is no more possible for the agents to continue doing so and hence any further monetary growth cause the value of money to decrease much faster than before. This dynamics between monetary growth rate, value of money and hours worked determine the consumption profile ($c_t^b = \pi_t \phi_t M_{t-1}$) of the central bank.

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