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**AN OVERALL MEASURE OF TECHNICAL INEFFICIENCY AT THE FIRM AND AT
THE INDUSTRY LEVEL: THE ‘LOST RETURN ON THE DOLLAR’ REVISITED**

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Abstract

As a measure of overall technical inefficiency the Directional Distance Function (DDF) introduced by Chambers, Chung, and Färe ties the potential output expansion and input contraction together through a single parameter. By duality, the DDF is related to a measure of profit inefficiency, which is calculated as the normalized deviation between optimal and actual profit at market prices. As we show, in the most usual case, the associated normalization represents the sum of the actual revenue and the actual cost of the assessed firm. Consequently, the corresponding dual formulation of the DDF has no obvious economic interpretation. In contrast, in this paper we allow outputs to expand and inputs to contract by different proportions. This results in a modified DDF that retains most of the properties of the original DDF. The corresponding dual problem has much simpler interpretation as the lost return on outlay that can be decomposed into a technical and an allocative inefficiency component.

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1. Introduction

A firm is considered to be technically inefficient if it is possible either to expand its output bundle without requiring any increase in its inputs or to contract its input bundle without requiring a reduction in its outputs. The potential for augmenting the output bundle reflects output-oriented inefficiency. Similarly, potential reduction in inputs signifies input-oriented inefficiency. In most empirical applications, technical efficiency is measured either in input- or in output-orientation. The choice between the two depends on the context. When the output bundle is a pre-assigned target (like the number of patients to be treated in a hospital), economizing on the use of inputs to contain costs is the objective. Therefore, attaining input-oriented technical efficiency by scaling down the input bundle to the extent possible is the logical first step. To attain full cost efficiency one would need to alter the input mix in light of the relative prices of inputs. Potential for cost reduction through input substitution relates to the allocative efficiency of the firm. In a comparable way, when inputs are predetermined (like land, labor, and farm yard manure in a family farm) generating the maximum revenue is the appropriate objective and attaining full output-oriented technical efficiency is more relevant. In textbook economics, the final objective of a firm in a competitive market economy is to maximize profit. With prices of both outputs and inputs parametrically given, the individual firm maximizes profit by selecting a feasible input-output bundle such that the difference between the corresponding revenue and cost is maximized. Profit maximization requires that at the optimal input-output choice, the input bundle chosen minimizes the cost of the corresponding output bundle and *at the same time* the output bundle generates the maximum possible revenue from the input bundle selected. In this case, there is no particular reason to select either the input- or output-oriented measure of technical efficiency. Profit maximization by an inefficient firm generally requires altering both the input and the output bundle. It is desirable, therefore, to use a technical efficiency measure that includes both an input-saving and an output-expanding component.

In the existing literature, there are several measures of technical efficiency that reflect output expansion and input contraction possibilities simultaneously. These include the Graph Hyperbolic measure (Färe et al., 1985), the Directional Distance Function (Chambers et al., 1998), the Enhanced Russell Graph measure (Pastor et al., 1999), the

Geometric Distance Function (Portela and Thanassoulis, 2005), the Generalized Efficiency measure (Ray, 2007), among others.

In the present paper we introduce a new measure of overall technical (in)efficiency that is similar to Ray (2007) but can be directly related to the Directional Distance Function. Also, we perform duality analysis to show that the new measure of technical efficiency provides a lower bound on the *lost return on the dollar*. An important contribution of this paper is to show how the proposed model can be used to measure overall technical efficiency at the industry level as simple average of inefficiencies at the firm level through a suitable choice of the direction for efficient projection of inefficient bundles onto the frontier. Our approach differs from a comparable model proposed in Leleu and Briec (2009) in that we allow the output expansion and input contraction scalars to be independent. Moreover, our normalization is different and yields a measure of technical efficiency that is a component of overall economic efficiency measured by the lost return on *the dollar spent* rather than on *the sum of dollars earned and dollars spent*. Finally, we point out that the new approach will be operationalized in a DEA (Data Envelopment Analysis) framework.

The rest of the paper is organized as follows. Section 2 provides a brief overview of the nonparametric DEA methodology and the Directional Distance Function. Section 3 introduces the new measure of overall technical inefficiency. Section 4 relates this new measure of technical inefficiency to the “lost return of outlay” through duality. Section 5 provides a numerical example. Section 6 is the summary.

2. The Nonparametric Methodology

2.1 The Technology and Technical Efficiency

Consider an industry producing bundles of s outputs y from bundles of m inputs x . An input-output bundle (x, y) constitutes a feasible production plan if output y can be produced from the input x . The production technology is defined by the production possibility set

$$T = \{(x, y) : y \in R_+^s \text{ can be produced from } x \in R_+^m\}. \quad (1)$$

Thus, an input-output bundle (x_0, y_0) is feasible, if $(x_0, y_0) \in T$.

The frontier of the production possibility set, often described as the *graph of the technology* is the set

$$G = \{(x, y) : (x, y) \in T, \beta < 1 \Rightarrow (\beta x, y) \notin T; \alpha > 1 \Rightarrow (x, \alpha y) \notin T\}. \quad (2)$$

This means that when an input-output bundle is in G it is not possible to increase *all* outputs without increasing any input or to reduce *all* inputs without reducing any output.

The bundle (x_0, y_0) is *weakly efficient* in its *input-orientation* if it is not possible to reduce all inputs simultaneously without reducing any output. That is,

$$(x_0, y_0) \in T \text{ and } \beta < 1 \Rightarrow (\beta x_0, y_0) \notin T. \quad (3)$$

Similarly, (x_0, y_0) is *weakly efficient* in its *output-orientation* if

$$(x_0, y_0) \in T \text{ and } \alpha > 1 \Rightarrow (x_0, \alpha y_0) \notin T. \quad (4)$$

That is all outputs cannot be increased simultaneously without increasing any input.

As defined by Farrell¹ (1957), the input-oriented technical efficiency of a firm producing output y_0 from input x_0 is measured as

$$\tau_x = \min \{\theta : (\theta x_0, y_0) \in T\}. \quad (5)$$

¹ Essentially the same concept was defined earlier as the *coefficient of resource utilization* by Debreu (1951) and as the (input) distance function by Shephard (1953).

A value of τ_x less than 1 implies that it is possible to reduce all inputs without lowering any output. In a parallel way, one can measure the output-oriented technical efficiency of the firm as

$$\tau_y = \min \left\{ \frac{1}{\phi} : (x_0, \phi y_0) \in T \right\}. \quad (6)$$

A value of τ_y less than 1 implies that there exists a factor ϕ greater than unity such that all outputs can be scaled up by this factor without requiring any increase in any input.

2.2. Data Envelopment Analysis (DEA)

In order to numerically measure the technical efficiency of a firm one needs to estimate the production possibility set empirically. In Stochastic Frontier Analysis (SFA) one specifies an explicit form of a production or cost function and uses econometric techniques to estimate the parameters of the specified model. In the nonparametric alternative approach of DEA introduced by Charnes, Cooper, and Rhodes (CCR) (1978) and further generalized to allow variable returns to scale by Banker, Charnes, and Cooper (BCC) (1984), one makes only a minimal set of fairly general assumptions about the underlying technology and constructs a piece-wise linear frontier that envelops the data using mathematical programming. In particular, one makes the following assumptions about T :

A1. Inputs are freely disposable. Thus, if $(x_0, y_0) \in T$ and $x_1 \geq x_0$, then $(x_1, y_0) \in T$.

A2. Outputs are freely disposable. Thus, if $(x_0, y_0) \in T$ and $y_1 \leq y_0$, then $(x_0, y_1) \in T$.

A3. The production possibility set is convex.

Suppose that data are available for the input-output quantities for n firms in an industry. Let $x_j \in R_+^m$ be the input and $y_j \in R_+^s$ be the output bundle of firm j ($j = 1, 2, \dots, n$). Then an empirical estimate of the smallest production possibility set T satisfying assumptions (A1-A3) is

$$S = \left\{ (x, y) \in R_+^{m+s} : x \geq \sum_{j=1}^n \lambda_j x_j, y \leq \sum_{j=1}^n \lambda_j y_j, \sum_{j=1}^n \lambda_j = 1; \lambda_j \geq 0, j = 1, 2, \dots, n \right\}. \quad (7)$$

The set S is also known as the free disposal convex hull of the observed input-output vectors.

Using S as the reference technology the input-oriented radial technical efficiency of a firm producing output y_0 from input x_0 is obtained by solving the input-oriented BCC DEA problem as

$$\begin{aligned} \hat{\tau}_x = \min \quad & \theta \\ \text{s.t.} \quad & \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{i0}, \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0}, \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \\ & \theta \text{ free} \end{aligned} \quad (8)$$

Similarly, the output-oriented radial efficiency is $\hat{\tau}_y = \frac{1}{\varphi^*}$ where

$$\begin{aligned} \varphi^* = \max \quad & \varphi \\ \text{s.t.} \quad & \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi y_{r0}, \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \\ & \varphi \text{ free} \end{aligned} \quad (9)$$

As noted above, the input-oriented model attributes all inefficiency to wastage in inputs. Similarly, in an output-oriented model all inefficiency is due to under achievement of potential output.

In a simple 1-output 1-input example if one measures output along the vertical axis and input along the horizontal axis in a 2-dimensional diagram, an output-oriented model projects the inefficient bundle vertically onto the frontier while the input-oriented model projects the same bundle horizontally towards the left. By contrast, the Directional Distance Function introduced by Chambers, Chung and Färe (CCF) (1998) allows the analyst to select the direction in which the inefficient bundle is projected onto the frontier. Let the vector $(g^x, g^y) \in R_+^{m+s}$, with $(g^x, g^y) \neq 0$, be any arbitrary direction in which the bundle (x_0, y_0) is to be projected. Then the corresponding efficient projection is

$$(x_0 - \beta g^x, y_0 + \beta g^y) \in G. \quad (10)$$

The scalar β is known as the Directional Distance Function (DDF). If one selects (x_0, y_0) for (g^x, g^y) , the corresponding efficient projection becomes $((1 - \beta)x_0, (1 + \beta)y_0)$. Suppose β equals 0.15. That would mean that at the directionally efficient projection input bundle is scaled down by 15% while the output bundle is scaled up by 15%. Clearly, β incorporates both input and output inefficiencies. The DEA problem for measuring the DDF is (see Färe and Grosskopf, 2000):

$$\begin{aligned} \bar{D}(x_0, y_0; g^x, g^y) = & \max \beta \\ \text{s.t.} & \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} - \beta g_i^x, \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} + \beta g_r^y, \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \\ & \beta \text{ free} \end{aligned} \quad (11)$$

It may be noted that although β is unrestricted in sign, because (x_0, y_0) is a feasible bundle, the maximum value of β will always be non-negative. One may note here that for $(g^x = x_0, g^y = 0)$ one gets the standard input-oriented problem while for $(g^x = 0, g^y = y_0)$ the problem in (11) reduces to the output-oriented model. Therefore, the DDF is a more general formulation with the usual input- and output-oriented models as special cases.

Although more flexible than the simple input- or output-oriented radial models, the DDF has its own limitations. In the first place, the rates of input reduction and output expansion are unnecessarily tied together. But a more serious problem is one of economic interpretation of the DDF that becomes apparent when one looks at the duality between the profit function and the DDF. As argued by CCF, an alternative characterization of the production possibility set is

$$T = \{(x, y) : \bar{D}(x, y; g^x, g^y) \geq 0\}. \quad (12)$$

Consider the dual profit function

$$\pi(p, w) = \max_{x, y} \left\{ \sum_{r=1}^s p_r y_r - \sum_{i=1}^m w_i x_i : (x, y) \in T \right\}. \quad (13)$$

It can be shown that (see Chambers *et al.* 1998, p. 358):

$$\bar{D}(x_0, y_0; g^x, g^y) \leq \frac{\pi(p, w) - \left(\sum_{r=1}^s p_r y_{r0} - \sum_{i=1}^m w_i x_{i0} \right)}{\sum_{r=1}^s p_r g_r^y + \sum_{i=1}^m w_i g_i^x}. \quad (14)$$

The numerator in the right hand side (RHS) expression is easily recognized to be the difference between the maximum attainable profit and the actual profit of the firm. In this sense, it is the lost profit due to inefficiency. For $(g^x = x_0, g^y = y_0)$, the most usual

selection of the reference vector g , the denominator is the sum of the actual revenue and the actual cost of the firm. In this sense, the RHS in (14) is not the standard ratio used as an economic index of the performance of a firm in order to evaluate the level of loss profit. Indeed, this expression has no obvious economic meaning. By implication, the profit inefficiency measure related to the DDF has no intuitive interpretation from a managerial point of view.

On the other hand, Ray (2007) addressed the problem of *shadow profit maximization* and through the dual problem (Ray 2007, p. 233) proposed the following overall measure of inefficiency:

$$\begin{aligned}
\delta = \max \quad & \varphi - \theta \\
s.t. \quad & \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{i0}, \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi y_{r0}, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \varphi, \theta \text{ free}
\end{aligned} \tag{15}$$

It is clear that with the added restrictions $\varphi = 1 + \beta$ and $\theta = 1 - \beta$, δ^* in the problem above would be reduced to $2\beta^*$, establishing the existing relationship between the Ray's measure and the DDF.

3. A New Measure of Overall Technical Inefficiency

In Ray (2007) the focus was on (shadow) profit maximization. Therefore increasing inputs was permissible if outputs could be increased sufficiently so that (shadow) profit would rise. A similar reasoning allowed reducing outputs alongside inputs. Increasing or decreasing inputs and outputs simultaneously involves tradeoffs between outputs and inputs. The conventional approach of allowing outputs to increase only and inputs to decrease only avoids such tradeoffs, which may be regarded as appropriate when the

objective is to measure technical efficiency only. Defining $\varphi = 1 + \beta^y$ and $\theta = 1 - \beta^x$ and imposing the restrictions $\varphi \geq 1$ and $\theta \leq 1$ we get the new measure of combined output- and input-oriented technical inefficiency from (15) as follows.

$$\begin{aligned}
\psi = \max \quad & \beta^x + \beta^y \\
s.t. \quad & \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} - \beta^x x_{i0}, \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} + \beta^y y_{r0}, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \beta^x, \beta^y \geq 0
\end{aligned} \tag{16}$$

Although a minor modification of (15), the LP problem in (16) can also be viewed as a modified Directional Distance Function:

$$\tilde{D}(x_0, y_0; x_0, y_0) = \max \left\{ \beta^x + \beta^y : (x_0 - \beta^x x_0, y_0 + \beta^y y_0) \in T, \beta^x, \beta^y \geq 0 \right\}. \tag{17}$$

The formulation in (17) allows the problem in (16) to break away from the overall inefficiency measure of Ray (2007) shown in (15) by selecting any arbitrary $(g^x, g^y) \in R_{++}^{m+s}$ that can differ from the observed bundle (x_0, y_0) . In general, the modified Directional Distance Function can be written as

$$\begin{aligned}
\vec{D}(x_0, y_0; g^x, g^y) = & \max \beta^x + \beta^y \\
& s.t. \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0} - \beta^x g_i^x, \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{r0} + \beta^y g_r^y, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \beta^x, \beta^y \geq 0
\end{aligned} \tag{18}$$

For example, one might select the sample average input-output bundle $\left(\bar{x} = \frac{1}{n} \sum_{j=1}^n x_j, \bar{y} = \frac{1}{n} \sum_{j=1}^n y_j \right)$ for (g^x, g^y) . For any firm k ($k = 1, 2, \dots, n$) the problem in (18) would then become

$$\begin{aligned}
\vec{D}(x_k, y_k; \bar{x}, \bar{y}) = & \max \beta^x + \beta^y \\
& s.t. \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ik} - \beta^x \bar{x}_i, \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rk} + \beta^y \bar{y}_r, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \beta^x, \beta^y \geq 0
\end{aligned} \tag{19}$$

The corresponding efficient projection would be

$$(x_k^*, y_k^*) = (x_k - \beta_k^{x*} \bar{x}, y_k + \beta_k^{y*} \bar{y}). \tag{20}$$

Note also that if $\vec{D}(x_0, y_0; g^x, g^y)$, the modified Directional Distance Function, is equal to zero, then $(x_0, y_0) \in G$. The original Directional Distance Function satisfies this sufficient condition on efficiency as well.

3.1. Measuring Inefficiency at the Industry Level

Aggregating over all firms, the technically efficient input-output bundle for the industry (keeping the existing distribution of inputs across firms) would be

$\left(X^* = \sum_{k=1}^n x_k^*, Y^* = \sum_{k=1}^n y_k^* \right)$. Note, however, that

$$\sum_{k=1}^n x_k^* = \sum_{k=1}^n x_k - \bar{x} \left(\sum_{k=1}^n \beta_k^{x*} \right) = \sum_{k=1}^n x_k \left(1 - \frac{1}{n} \sum_{k=1}^n \beta_k^{y*} \right) \quad (21)$$

Similarly,

$$\sum_{k=1}^n y_k^* = \sum_{k=1}^n y_k \left(1 + \frac{1}{n} \sum_{k=1}^n \beta_k^{y*} \right) \quad (22)$$

Define,

$$X = \sum_{k=1}^n x_k, Y = \sum_{k=1}^n y_k, \bar{\beta}^{x*} = \frac{1}{n} \sum_{k=1}^n \beta_k^{x*}, \bar{\beta}^{y*} = \frac{1}{n} \sum_{k=1}^n \beta_k^{y*}. \quad (23)$$

Then

$$X^* = (1 - \bar{\beta}^{x*}) X; Y^* = (1 + \bar{\beta}^{y*}) Y. \quad (24)$$

This means that $\bar{\beta}^{x*}$ measures the input inefficiency and $\bar{\beta}^{y*}$ the output inefficiency *for the industry as a whole*.

It is important to distinguish between the DEA problems in (16) and (19). The problem in (16) is a direct measure of the inefficiency of the individual firm because the benchmark input bundle is proportional to the actual input and the benchmark output bundle is proportional to the actual output bundle of the firm. By contrast, the

benchmark in (19) is *not proportional to the actual input and output bundles* of the firm under evaluation. Hence, it is not really a measure of the inefficiency of the firm in the usual sense. But the usefulness of (19) lies in the fact that when averaged across all the firms, it yields the *actual* inefficiency of the entire industry.

It should be noted that Leleu and Briec (LB) (2009) proposed a model that is very similar to the model in (19). The LB model can be written as

$$\begin{aligned}
\vec{D}(x_k, y_k; X, Y) = & \max \beta_k \\
& s.t. \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{ik} - \beta_k X_i, \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rk} + \beta_k Y_r, \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \beta_k \text{ free}
\end{aligned} \tag{25}$$

It is clear that the LB model is more restrictive than (19) since they use the same scalar, β_k , for changing inputs and outputs. Further, being another example of the CCF Directional Distance Function, their dual formulation will have the normalization by the *sum* of the total revenue and the total cost of the industry.

3.2. Relation with the Directional Distance Function

Although we have described the new measure in (18) (of which (17) is a special case) as a modified Directional Distance Function, it does not have all the properties listed in Lemma 2.2 of Chambers *et al.* (1998, p. 356)². But if we consider $(\lambda^*, \beta^{x*}, \beta^{y*})$ as an optimal solution of (18) and assume, without loss of generality, that $\beta^{y*} > 0$ and $\beta^{y*} > \beta^{x*}$ and defining $\mu = \frac{\beta^{x*}}{\beta^{y*}}$ and $\tilde{g}^x = \mu g^x$, then (18) can be written as

² The modified Directional Distance Function satisfies all the properties in Lemma 2.2 but property (a). In particular, $\vec{D}(x_0 - \alpha g^x, y_0 + \alpha g^y; g^x, g^y) = \vec{D}(x_0, y_0; g^x, g^y) - 2\alpha$.

$\vec{D}(x_0, y_0; g^x, g^y) = (1 + \mu) \bar{D}(x_0, y_0; \tilde{g}^x, \tilde{g}^y)$. Therefore, the modified Directional Distance Function is indeed a DDF scaled by a known factor $(1 + \mu)$.

4. Duality and the decomposition of the lost return on outlay

In for-profit organizations, where we usually have not only quantity but also market price information, the measurement of profit inefficiency is particularly important and, in fact, is generally the most important objective of the firm. Specifically, profit inefficiency reflects the comparison of the observed profit of the firm under evaluation with the maximum attainable at the given market prices and the technology. Usually in the existing literature, the profit inefficiency has been decomposed into two components: technical and allocative inefficiencies. Technical inefficiency measures how close the firm is to the frontier of the technology and has nothing to do with market prices. By contrast, given market prices, allocative inefficiency for technically efficient units measures the loss due to not using the optimal marginal rate of substitution between inputs and outputs.

Regarding the measurement and the decomposition of the profit inefficiency, it is worth mentioning that duality between the technical efficiency measure and the profit function has been the most suitable tool for defining overall performance measures and decomposing them. Duality allows relating the optimal value of two optimization programs, one corresponding to the determination of the optimal profit and the other associated with the calculation of the technical efficiency. In this respect, the best known in the literature is the DDF approach. By duality, the DDF is related to the profit function through a particular normalization condition on prices. Specifically as we showed in (14), if we consider the usual reference vector $(g^x, g^y) = (x_0, y_0)$ then

$$\bar{D}(x_0, y_0; x_0, y_0) \leq \frac{\pi(p, w) - \pi_0}{R_0 + C_0},$$

where π_0 , R_0 and C_0 are, respectively, the firm's

actual profit, revenue and cost at market prices. However, in this case, the profit inefficiency measure (the RHS) presents some problems with respect to its economic interpretation, as we argued at the end of Section 2.2.

Overall, in the literature there exist two different ways to assess the economic loss due to profit inefficiency. On the one hand, we could use a ratio-form measure as $\pi_0/\pi(p, w)$. Unfortunately, if $\pi(p, w)$ is zero then this ratio becomes ill-defined (see, for example, Portela and Thanassoulis, 2007). A second alternative to evaluate profit inefficiency involves the use of a difference-form measure as $\pi(p, w) - \pi_0$. In particular, this is the case of the DDF approach. The difference-form measure takes only non-negative values and, particularly, a value of zero is related to nil inefficiency. However, as Nerlove (1965, p. 94) pointed out, this alternative has one serious shortcoming. In particular, it is not an appropriate economic measure because it is homogeneous of degree one in prices. One solution of this problem was proposed by Chambers et al. (1998) through the DDF. Specifically, using $\pi(p, w) - \pi_0$ and a deflator (the sum of the actual revenue and the actual cost of the assessed firm) the measure meets the homogeneity property of degree zero in prices. Unfortunately, the underlying problem is then the interpretability of the measure for the manager of the firm.

Something similar happens with a recent approach introduced by Cooper et al. (2011a). In that paper, the authors are able to derive a duality result relating the Weighted Additive measure (see, for example, Cooper et al., 1999 and Cooper et al., 2011b) in DEA to the lost profit due to inefficiency. In particular, they proved that

$$WA(x_0, y_0; b^x, b^y) \leq \frac{\pi(p, w) - \pi_0}{\min \left\{ \frac{w_1}{b_1^x}, \dots, \frac{p_s}{b_s^y} \right\}}, \quad (26)$$

where $WA(x_0, y_0; b^x, b^y)$ denotes the Weighted Additive measure given strictly positive weights for inputs and outputs (b^x, b^y) . In other words, the Weighted Additive measure is a lower bound of the overall inefficiency. In this way, they proposed a new profit inefficiency measure, the RHS in (26), normalized by a particular condition: the minimum among the ratios of market prices to weights. Obviously, the economic interpretation of this condition is not clear again. Nevertheless, in both approaches (the DDF and the WA measure) the overall inefficiency measure satisfies a very interesting

set of index number properties (see Kuosmanen et al., 2010). Specifically, they are homogeneous of degree zero in prices and quantities, and it is also well-defined for not positive profits.

Other related approaches are the ones by Cooper et al. (1999) and that by Portela and Thanassoulis (2007). On the one hand, Cooper et al. (1999) suggest the expression $\pi(p, w) - \pi_0$ (without deflator) as the profit inefficiency measure and decompose the overall waste into a technical component that is estimated through the original additive model (Charnes et al., 1985). However, this approach presents two drawbacks. First, the profit inefficiency measure is homogeneous of degree one in prices. And second, the value of the technical component is not independent of alternative optimal solutions of the additive model (see Cooper et al., 2011a, p. 412). On the other hand, Portela and Thanassoulis (2007) propose to use the Geometric Distance Function in order to decompose a new measure of profitability that is based in the following ratio.

$$\frac{\left(\prod_{r=1}^s p_r y_{r0} \right)^{1/s}}{\left(\prod_{i=1}^m w_i x_{i0} \right)^{1/m}} . \quad (27)$$

Although the Geometric Distance Function approach satisfies several interesting properties, it is obvious that (27) is really difficult to interpret in economics terms from a managerial point of view.

In view of the preceding discussion, it seems necessary to find a new approach that relates the lost profit inefficiency to the technical inefficiency component using a suitable normalization condition. Suitable in the sense that it has to meet several desirable properties and also the normalized overall inefficiency measure should make sense in economics terms. Our approach allows deriving such type of relationship in a natural way by simply applying duality, as we next show.

First of all, in order to reach our goals we need to show the dual problem of (18), which is (28).

$$\begin{aligned}
\text{Min} \quad & \alpha_0 - \sum_{r=1}^s u_{r0} y_{r0} + \sum_{i=1}^m v_{i0} x_{i0} \\
\text{s.t.} \quad & \alpha_0 - \sum_{r=1}^s u_{r0} y_{rj} + \sum_{i=1}^m v_{i0} x_{ij} \geq 0, \quad j = 1, \dots, n \\
& \sum_{i=1}^m v_{i0} g_i^x \geq 1, \\
& \sum_{r=1}^s u_{r0} g_r^y \geq 1, \\
& u_{r0} \geq 0, \quad r = 1, \dots, s \\
& v_{i0} \geq 0, \quad i = 1, \dots, m \\
& \alpha_0 \text{ free}
\end{aligned} \tag{28}$$

Obviously, the optimal value of (28) coincides with $\tilde{D}(x_0, y_0; g^x, g^y)$. In addition, let $(u_0^*, v_0^*, \alpha_0^*)$ be an optimal solution of (28), then u_0^* and v_0^* can be interpreted as input and output shadow prices, respectively, and $\alpha_0^* = \pi(u_0^*, v_0^*)$. In other words, α_0^* represents shadow profit for the evaluated firm with bundle (x_0, y_0) .

Next we show an equivalent way of writing problem (28), which will allow us to establish a new Fenchel-Mahler inequality (see Cooper et al., 2011a) between the optimal profit and the modified Directional Distance Function. Indeed, given strictly positive reference vectors, g^x and g^y , Proposition 1 shows how to derive $\tilde{D}(x_0, y_0; g^x, g^y)$ from the profit function.

Proposition 1. For a firm with bundle (x_0, y_0) , $g^x > 0_m$ and $g^y > 0_s$, $\tilde{D}(x_0, y_0; g^x, g^y)$ is equal to the optimal value of the following optimization program.

$$\begin{aligned}
& \text{Inf} \quad \frac{\pi(u_0, v_0) - \left(\sum_{r=1}^s u_{r0} y_{r0} - \sum_{i=1}^m v_{i0} x_{i0} \right)}{\min \left\{ \sum_{i=1}^m v_{i0} g_i^x, \sum_{r=1}^s u_{r0} g_r^y \right\}} \\
& \text{s.t.} \\
& \sum_{i=1}^m v_{i0} > 0, \\
& \sum_{r=1}^s u_{r0} > 0, \\
& v_{i0} \geq 0, \quad i = 1, \dots, m \\
& u_{r0} \geq 0, \quad r = 1, \dots, s
\end{aligned} \tag{29}$$

Proof. See Appendix.

In order to develop an appropriate overall inefficiency measure, let us consider the sample average input-output bundle $(\bar{x}, \bar{y})$ for (g^x, g^y) . Then, applying Proposition 1, we derive an easily interpretable result which establishes that lost return on the average outlay is directly related to the modified Directional Distance Function.

Corollary 1. Let $w \in R_{++}^m$ and $p \in R_{++}^s$ be input and output market prices, respectively. Let $g^x = \bar{x} > 0$ and $g^y = \bar{y} > 0$. Then, for an industry with non-negative aggregated profit, i.e., $\sum_{k=1}^n \pi_k \geq 0$, the following inequality holds:

$$\frac{\pi(p, w) - \pi_0}{\bar{C}} \geq \tilde{D}(x_0, y_0; \bar{x}, \bar{y}), \tag{30}$$

where $\bar{C} = \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^m w_i x_{ik} = \frac{1}{n} \sum_{k=1}^n C_k$, i.e., the sample average cost.

Proof. See Appendix.

The expression on the left of (30) provides a measure of the profit lost by not operating in a fully efficient manner. Specifically, this measure is a normalized deviation between

optimal profit and observed profit for the evaluated firm at market prices. Unlike the Directional Distance Function and the Weighted Additive measure approaches, in this case the normalization factor is clearly interpretable: the average cost of the whole industry. It means that the overall inefficiency measure we propose is directly the well-known lost return on outlay (see Ray, 2004, p. 233-234). However, while Ray related the lost return on outlay of the firm to an input-oriented measure, neglecting the output-oriented inefficiencies, we propose as technical inefficiency component a non-oriented measure, $\tilde{D}(x_0, y_0; \bar{x}, \bar{y})$, which takes into account input contraction and output expansion simultaneously.

Regarding the properties that the overall inefficiency measure should meet, it is easy to prove that the lost returns on the average outlay is homogeneous of degree zero in prices and quantities, and it is well-defined for non-positive profit. In addition, this measure is non-negative with nil inefficiency signalled when it takes value zero. Indeed, the measure satisfies the condition that it is zero if and only if the unit being assessed achieves maximum profit, i.e., it meets indication (see Portela and Thanassoulis, 2007).

In the Farrell (1957) tradition, we define allocative inefficiency (AI) as the residual that allows us to close the inequality in (30), providing us with the decomposition of the lost return on average outlay into technical and allocative components:

$$\underbrace{\frac{\pi(p, w) - \pi_0}{\bar{C}}}_{\text{Overall Inefficiency}} = \underbrace{\tilde{D}(x_0, y_0; \bar{x}, \bar{y})}_{\text{Technical Inefficiency}} + AI_0. \quad (31)$$

Overall, the propose measure evaluates the lost profit through the average cost of the whole industry. In contrast, if we wish to measure the overall inefficiency for each firm in terms of its particular lost return on outlay, then we can use the following identity:

$$\frac{\pi(p, w) - \pi_0}{C_0} = \frac{\pi(p, w) - \pi_0}{\bar{C}} \cdot \frac{\bar{C}}{C_0}, \quad (32)$$

where the last expression, $\bar{C}/C_0$, can be interpreted as a size effect. In this way, the lost return on (individual) outlay maybe decomposed into the lost return on the average outlay and a measure of firm size effect on the whole industry.

We also want to highlight that we can define an overall inefficiency measure for the whole industry as the mean of the individual overall inefficiencies. If we proceed in this way we get as index the average lost return on the average outlay:

$$\frac{1}{n} \sum_{j=1}^n \frac{\pi(p, w) - \pi_j}{\bar{C}} = \frac{\pi(p, w) - \bar{\pi}}{\bar{C}}, \quad (33)$$

which can be easily decomposed into technical inefficiency as $\frac{1}{n} \sum_{j=1}^n \tilde{D}(x_j, y_j; \bar{x}, \bar{y})$ and allocative inefficiency as $\frac{1}{n} \sum_{j=1}^n AI_j$.

Finally, we point out that one potential problem with the technically efficient projection $(x_0 - \beta^x \bar{x}, y_0 + \beta^y \bar{y})$ of the actual input-output bundle (x_0, y_0) is that input and/or output slacks may remain at this point on the frontier. However, incorporating slacks into a scalar measure of efficiency requires either non-radial movement or assigning prior weights to the individual inputs and outputs. When measuring technical efficiency is an end in itself, a non-radial measure (like the Enhanced Russell Graph measure of Pastor et al, 1999, or the Geometric Distance Function of Portela and Thanassoulis, 2007) could be used. However, when one measures technical efficiency as a component of cost, revenue, or profit efficiency, it is best to incorporate the slacks in the residual allocative component. It is well known that no slacks can exist at the optimal solution of the profit maximization problem. In other words, the final solution is always Pareto-Koopmans efficient. But, if one uses weights for slacks in measuring technical efficiency, there arises a problem of inconsistency if the weights are not proportional to the prices. On the other hand, if the weights reflect the market prices, the technical efficiency measure becomes price dependent. We believe that it is best to keep the technical efficiency radial and to incorporate possible slacks in allocative efficiency.

5. Numerical Example

In this section, we provide an application of the proposed measure using the input and output price and quantity data from Banker and Maindiratta (1988) for this illustration. The original data was obtained from a division of a large decentralized U.S. manufacturing firm. Specifically, the data consist of 20 quarterly observations on quantities and prices of one output and three inputs (labor, material and capital) for the period from 1979 to 1983.

A Pareto-efficiency analysis using the additive model (Charnes et al., 1985) revealed nine units as efficient (the maximum sum of input and output slacks is zero): units 2, 5, 6, 12, 13, 14, 15, 17 and 20. For this particular database, the modified DDF detects the same observations as efficient with an optimal value of zero for technical overall inefficiency. Table 1 reports for all units in the sample the results obtained when we maximize the sum of β^x and β^y considering $(g^x, g^y) = (x_0, y_0)$ and $(g^x, g^y) = (\bar{x}, \bar{y})$ as reference vectors. For each selection of g we show β^{x*} and β^{y*} , the value of the modified DDF and the peer group of efficient units used to obtain the target of each observation onto the frontier.

<Insert Table 1 approximately here>

The first thing that needs to be highlighted from Table 1 is that the results are very similar regardless the considered reference vector. Also, among the inefficient units, only for unit 19 the inefficiency is entirely on the output side. For the remaining inefficient units the technical inefficiency is attributable to improvement only in the input side, for units 1, 4, 8, 9 and 10, or enhancements in inputs and outputs at the same time, for units 7, 11 and 16. To be specific, we may look, for instance, at the results for the inefficient unit 19. This unit should increase its output by 3.12% if we consider $(g^x, g^y) = (x_0, y_0)$ or by 2.74% respect to the output mean in the sample, $\bar{y} = 86691.7$, i.e., 2375.35 units, if we consider $(g^x, g^y) = (\bar{x}, \bar{y})$ in order to achieve the efficiency. A unit with a different behaviour is, for example, unit 1. In this particular case, the unit should reduce each input by 3.82% if $(g^x, g^y) = (x_0, y_0)$ or by 3.88% regarding to the

corresponding input mean ($\bar{x}_1 = 28381.42$, $\bar{x}_2 = 36857.51$ and $\bar{x}_3 = 8075.69$) if $(g^x, g^y) = (\bar{x}, \bar{y})$. In other words, using $(g^x, g^y) = (\bar{x}, \bar{y})$, unit 1 has to decrease 1101.20, 1430.06 and 313.34 units in labor, material and capital, respectively. Finally, regarding the peers, the results show that units 2, 12, 13 and 17 are the most relevant benchmarks in the evaluation of the inefficient units.

<Insert Table 2 approximately here>

While Table 1 relates to technical inefficiency, the results regarding profit inefficiency and its decomposition are shown in Table 2. For each of the units in the sample, we have reported the actual profit, the optimal profit, the actual cost, the lost return on individual outlay (using C_0 for each observation), the lost return on average outlay (using $\bar{C} = 79981.92$ for each observation), the size effect, i.e., the ratio $\bar{C}/C_0$, and, finally, the components of the overall inefficiency, i.e., the technical inefficiency, measured through the modified DDF for $(g^x, g^y) = (\bar{x}, \bar{y})$, and the allocative inefficiency retrieved as a residual. As for these two sources of inefficiency, we have also reported, between parentheses, the percentages with respect to the lost return on average outlay (column 6). Looking at the results, the analysis reveals that only two units are fully efficient, in the sense that they are the targets points when profit at market prices is maximized. In particular, we are referring to units 2 and 13. Note that for these units, actual and optimal profits coincide and, consequently, overall inefficiency is zero. On the other hand, the remaining units are profit inefficient since optimal profit is strictly greater than the observed profit at market prices. Such overall inefficiency is reflected in both the lost return on individual outlay and the lost return on the average outlay through strictly positive values. In fact, the values obtained through both measures are very similar. To be precise, see, for instance, the results regarding unit 1. In this particular case, the amount of unrealized profit is $23642 - 17633 = 6009$, implying that lost return on individual outlay is equal to 0.0781, figures very similar to the lost return on average outlay, 0.0751. It means that this unit could earn 7.8% of its own actual cost and 7.5% of the cost of the average input bundle. Regarding the technical and allocative sources of inefficiency, 52% is due to technical inefficiency and 48% is due to allocative inefficiency. Also, we observe that the unit 1 is not a *big* unit in

the sample since its corresponding size effect is greater than one. Given the relationship in (32), we can explain the value of the lost return on outlay for the unit 1 as the lost return on the average outlay times a *negative* size effect. Negative in the sense that the fewer of lost return on outlay, the better, and a size effect greater than one increases the estimated value for the lost return on average outlay. Overall, in the sample the profit inefficiency is caused by only allocative reasons (units 5, 6, 12, 14, 15, 17 and 20) or due to the existence of both kinds of inefficiency, technical and allocative, in the case of units 1, 3, 4, 7, 8, 9, 10, 11, 16, 18 and 19.

Finally, we notice that the mean of the lost return on outlay over all the units is equal to 0.1388 and respect to the average outlay equals 0.1387. These similar values can be interpreted as the estimation of the overall inefficiency of the whole industry. Moreover, the lost return on the average outlay (0.1387) is due to technical inefficiency (45% or 0.0619) and allocative inefficiency (55% or 0.0768).

6. Conclusion

Profit maximization requires appropriate choice of both input and output quantities. When the actual input-output bundle of a firm is below the frontier, projecting it to a point on the frontier eliminates technical inefficiency. This paper proposes a new technical inefficiency measure that contracts the input bundle and expands the output bundle of a firm at different rates through a DEA problem. In contrast to other approaches like that derived from the Directional Distance Function, the lost return on outlay that corresponds to the dual of the DEA problem we propose has a clear economic meaning from a managerial point of view. Accordingly, as far as we are aware, we present the first example of a profit inefficiency measure that can be decomposed and economically interpreted in an easy way. In addition, use of the industry average of input-output bundle to define the direction of efficient projection lets us derive the industry efficiency as a simple average of firm efficiencies.

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Appendix

Proof of Proposition 1: We consider the following auxiliary optimization program.

$$\begin{aligned}
& \text{Inf} \quad \pi(u_0, v_0) - \left(\sum_{r=1}^s u_{r0} y_{r0} - \sum_{i=1}^m v_{i0} x_{i0} \right) \\
& \text{s.t.} \quad \min \left\{ \sum_{i=1}^m v_{i0} g_i^x, \sum_{r=1}^s u_{r0} g_r^y \right\} \geq 1 \\
& \quad \sum_{i=1}^m v_{i0} > 0, \\
& \quad \sum_{r=1}^s u_{r0} > 0, \\
& \quad v_{i0} \geq 0, \quad i = 1, \dots, m \\
& \quad u_{r0} \geq 0, \quad r = 1, \dots, s
\end{aligned} \tag{34}$$

We first prove that the optimal value of (34) equals $\tilde{D}(x_0, y_0; g^x, g^y)$. Let $(\tilde{u}_0, \tilde{v}_0)$ be a feasible solution of (34). Then, we start showing that $(\tilde{u}_0, \tilde{v}_0, \tilde{\alpha}_0)$ is a feasible solution for (28), considering $\tilde{\alpha}_0 = \pi(\tilde{u}_0, \tilde{v}_0)$. Thanks to the first constraint in (34) we have that

$$\sum_{i=1}^m \tilde{v}_{i0} g_i^x \geq 1 \text{ and } \sum_{r=1}^s \tilde{u}_{r0} g_r^y \geq 1. \text{ Also, since by definition } \pi(\tilde{u}_0, \tilde{v}_0) \geq \sum_{r=1}^s \tilde{u}_{r0} y_{rj} - \sum_{i=1}^m \tilde{v}_{i0} x_{ij},$$

for all $j = 1, \dots, n$, we have that $\tilde{\alpha}_0 - \sum_{r=1}^s \tilde{u}_{r0} y_{rj} + \sum_{i=1}^m \tilde{v}_{i0} x_{ij} \geq 0$, for all $j = 1, \dots, n$. Hence,

$(\tilde{u}_0, \tilde{v}_0, \tilde{\alpha}_0)$ is a feasible solution of (28). In addition, the objective function value in (28)

evaluated at point $(\tilde{u}_0, \tilde{v}_0, \tilde{\alpha}_0)$ is equal to $\tilde{\alpha}_0 - \sum_{r=1}^s \tilde{u}_{r0} y_{r0} + \sum_{i=1}^m \tilde{v}_{i0} x_{i0} =$

$$\pi(\tilde{u}_0, \tilde{v}_0) - \left(\sum_{r=1}^s \tilde{u}_{r0} y_{r0} - \sum_{i=1}^m \tilde{v}_{i0} x_{i0} \right), \text{ which is the objective function value in (34)}$$

evaluated at point $(\tilde{u}_0, \tilde{v}_0)$. As a consequence, for any feasible solution $(\tilde{u}_0, \tilde{v}_0)$ of (34)

$$\pi(\tilde{u}_0, \tilde{v}_0) - \left(\sum_{r=1}^s \tilde{u}_{r0} y_{r0} - \sum_{i=1}^m \tilde{v}_{i0} x_{i0} \right) \geq \tilde{D}(x_0, y_0; g^x, g^y). \text{ In other words, } \tilde{D}(x_0, y_0; g^x, g^y)$$

is a lower bound of the optimal value of (34). On the other hand, let $(u_0^*, v_0^*, \alpha_0^*)$ be an

optimal solution for (28). Let us see that then (u_0^*, v_0^*) is a feasible solution for (34).

First, $\sum_{i=1}^m v_{i0}^* g_i^x \geq 1$ and $\sum_{r=1}^s u_{r0}^* g_r^y \geq 1$ imply that $\min \left\{ \sum_{i=1}^m v_{i0}^* g_i^x, \sum_{r=1}^s u_{r0}^* g_r^y \right\} \geq 1$. Secondly, since $g^x > 0_m$ and $g^y > 0_s$, we have that $\exists i' = 1, \dots, m$ such that $v_{i'0}^* > 0$ and $\exists r' = 1, \dots, s$ such that $u_{r'0}^* > 0$. In this way, $\sum_{i=1}^m v_{i0}^* > 0$ and $\sum_{r=1}^s u_{r0}^* > 0$ since $v_{i0}^* \geq 0, i = 1, \dots, m$, and $u_{r0}^* \geq 0, r = 1, \dots, s$. As a consequence, $\pi(u_0^*, v_0^*) - \left(\sum_{r=1}^s u_{r0}^* y_{r0} - \sum_{i=1}^m v_{i0}^* x_{i0} \right)$ is greater or equal than the infimum of (34). Finally, applying that $\alpha_0^* = \pi(u_0^*, v_0^*)$, we have that

$$\pi(u_0^*, v_0^*) - \left(\sum_{r=1}^s u_{r0}^* y_{r0} - \sum_{i=1}^m v_{i0}^* x_{i0} \right) = \sum_{i=1}^m v_{i0}^* x_{i0} - \sum_{r=1}^s u_{r0}^* y_{r0} + \alpha_0^* = \vec{D}(x_0, y_0; g^x, g^y).$$

Hence, by the definition of infimum, we have that the optimal value of (28) equals the optimal value of (34). Next we show that the optimal value of (34) and (29) are equal. In order to do that, we are going to apply Lemma 2 in Cooper et al. (2011a, p. 413).

First, $Z = \left\{ (u_0, v_0) : \sum_{i=1}^m v_{i0} > 0, \sum_{r=1}^s u_{r0} > 0, v_{i0} \geq 0, i = 1, \dots, m, u_{r0} \geq 0, r = 1, \dots, s \right\}$ is a cone in R^{m+s} (see Rockafellar, 1970). Also, let us define $f(u_0, v_0) := \pi(u_0, v_0) - \left(\sum_{r=1}^s u_{r0} y_{r0} - \sum_{i=1}^m v_{i0} x_{i0} \right)$ and $h(u_0, v_0) := \min \left\{ \sum_{i=1}^m v_{i0} g_i^x, \sum_{r=1}^s u_{r0} g_r^y \right\}$. We observe that $\pi(u_0, v_0)$ is homogeneous of degree +1 in (u_0, v_0) (see Färe and Primont, 1995). Consequently, we have that $f(u_0, v_0)$ is also homogeneous of degree +1 in (u_0, v_0) . Moreover, $f(u_0, v_0) \geq 0$ since $(x_0, y_0) \in T$. On the other hand, $h(u_0, v_0)$ is strictly positive in Z and is also homogeneous of degree +1 in (u_0, v_0) . Then, by Lemma 2 in Cooper et al. (2011a), (34) is equivalent to (29). ■

Proof of Corollary 1: By hypothesis $\sum_{k=1}^n \pi_k \geq 0$, which implies that

$$\min \left\{ \sum_{i=1}^m w_i \frac{1}{n} \sum_{j=1}^n x_{ij}, \sum_{r=1}^s p_r \frac{1}{n} \sum_{j=1}^n y_{rj} \right\} = \sum_{i=1}^m w_i \frac{1}{n} \sum_{k=1}^n x_{ik} = \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^m w_i x_{ik} = \bar{C}. \text{ On the other}$$

hand, it is apparent that the vector (p, w) is a feasible solution for (29). Then, applying

Proposition 1, we have that $\frac{\pi(p, w) - \pi_0}{\bar{C}} \geq \vec{D}(x_0, y_0; \bar{x}, \bar{y})$. ■

Table 1. Technical inefficiency using the Modified Directional Distance Function for the numerical example

	$g = (x_0, y_0)$				$g = (\bar{x}, \bar{y})$			
Unit	β^{x*}	β^{y*}	$TI (\beta^{x*} + \beta^{y*})$	Peers	β^{x*}	β^{y*}	$TI (\beta^{x*} + \beta^{y*})$	Peers
1	0.0382	0.0000	0.0382	2 and 13	0.0388	0.0000	0.0388	2, 13 and 17
2	0.0000	0.0000	0.0000	2	0.0000	0.0000	0.0000	2
3	0.0009	0.1346	0.1354	12 and 13	0.0007	0.1193	0.1200	12 and 13
4	0.0971	0.0000	0.0971	2, 13 and 17	0.1047	0.0000	0.1047	2, 13 and 17
5	0.0000	0.0000	0.0000	5	0.0000	0.0000	0.0000	5
6	0.0000	0.0000	0.0000	6	0.0000	0.0000	0.0000	6
7	0.0471	0.0639	0.1111	12 and 13	0.0432	0.0601	0.1033	12 and 13
8	0.1081	0.0000	0.1081	2, 12 and 13	0.1174	0.0000	0.1174	2, 12 and 13
9	0.1587	0.0000	0.1587	2, 13 and 17	0.1749	0.0000	0.1749	2, 13 and 17
10	0.1428	0.0000	0.1428	2, 12 and 13	0.1524	0.0000	0.1524	2, 12 and 13
11	0.0392	0.1561	0.1953	12 and 13	0.0356	0.1348	0.1704	12 and 13
12	0.0000	0.0000	0.0000	12	0.0000	0.0000	0.0000	12
13	0.0000	0.0000	0.0000	13	0.0000	0.0000	0.0000	13
14	0.0000	0.0000	0.0000	14	0.0000	0.0000	0.0000	14
15	0.0000	0.0000	0.0000	15	0.0000	0.0000	0.0000	15
16	0.1339	0.0111	0.1450	13	0.1351	0.0110	0.1462	13
17	0.0000	0.0000	0.0000	17	0.0000	0.0000	0.0000	17
18	0.0519	0.0360	0.0879	13	0.0471	0.0355	0.0826	13 and 17
19	0.0000	0.0312	0.0312	13 and 20	0.0000	0.0274	0.0274	13 and 20
20	0.0000	0.0000	0.0000	20	0.0000	0.0000	0.0000	20

Table 2. Profit inefficiency and its decomposition for the numerical example

Unit	Actual profit	Optimal profit	Actual cost	Lost return on individual outlay	Lost return on average outlay	Size effect	Technical Inefficiency	Allocative Inefficiency
1	17633.00	23642.00	76960.00	0.0781	0.0751	1.0393	0.0388 (52%)	0.0363 (48%)
2	23642.00	23642.00	72279.00	0.0000	0.0000	1.1066	0.0000 (-)	0.0000 (-)
3	11514.00	23642.00	65338.00	0.1856	0.1516	1.2241	0.1200 (79%)	0.0316 (21%)
4	13353.03	24200.34	83612.20	0.1297	0.1356	0.9566	0.1047 (77%)	0.0309 (23%)
5	18989.84	24199.77	86206.12	0.0604	0.0651	0.9278	0.0000 (0%)	0.0651 (100%)
6	21117.12	24200.91	82234.11	0.0375	0.0386	0.9726	0.0000 (0%)	0.0386 (100%)
7	13603.65	24199.49	70604.00	0.1501	0.1325	1.1328	0.1033 (78%)	0.0292 (22%)
8	14373.53	24464.53	86680.02	0.1164	0.1262	0.9227	0.1174 (93%)	0.0088 (7%)
9	7861.25	24458.29	89198.12	0.1861	0.2075	0.8967	0.1749 (84%)	0.0326 (16%)
10	12142.55	24469.64	84236.07	0.1463	0.1541	0.9495	0.1524 (99%)	0.0018 (1%)
11	6261.01	24448.36	73424.03	0.2477	0.2274	1.0893	0.1704 (75%)	0.0570 (25%)
12	18195.08	23651.20	76054.02	0.0717	0.0682	1.0516	0.0000 (0%)	0.0682 (100%)
13	23666.08	23666.08	72472.82	0.0000	0.0000	1.1036	0.0000 (-)	0.0000 (-)
14	6375.86	23697.33	82140.04	0.2109	0.2166	0.9737	0.0000 (0%)	0.2166 (100%)
15	424.98	23736.76	71084.92	0.3279	0.2915	1.1252	0.0000 (0%)	0.2915 (100%)
16	3321.40	23266.49	93926.98	0.2123	0.2494	0.8515	0.1462 (59%)	0.1032 (41%)
17	10856.71	23296.25	95404.04	0.1304	0.1555	0.8383	0.0000 (0%)	0.1555 (100%)
18	9703.13	23334.44	85202.99	0.1600	0.1704	0.9387	0.0826 (48%)	0.0878 (52%)
19	10113.02	23301.95	75584.98	0.1745	0.1649	1.0582	0.0274 (17%)	0.1375 (83%)
20	13782.47	25355.11	76995.98	0.1503	0.1447	1.0388	0.0000 (0%)	0.1447 (100%)