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Nonparametric Measures of Efficiency in the Presence of Undesirable Outputs: A By-production Approach with Weak Disposability

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Nonparametric Measures of Efficiency in the Presence of Undesirable Outputs:

A By-production Approach with Weak Disposability

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In empirical research on productivity measurement adjusted for undesirable outputs on the side, the good and the bad outcomes are treated as joint-products of the underlying production process. In the present paper, following Murty, Russell, and Levkoff, we conceptualize the good output as technologically separable from the bad output. Moreover, we set up an integrated DEA optimization problem over the intersection of these two sub-technologies to measure the efficiency of a firm that produces a bad output alongside the good output. In an empirical illustration of our methodology, we use country level data for an unbalanced panel of 64 countries over the years 1986 through 2011 where per capita GDP is the good and per capita CO_2 emission is the bad output. Weak disposability and null jointness is assumed between the bad output and fuel, the polluting input, rather than the good and bad outputs. We then utilize our DEA results to compute opportunity costs of a targeted reduction in CO_2 emission in terms of required dollar amounts of reduction in per capita GDP for the individual countries in selected years.

Keywords: Bad Output; Weak Disposability; Null jointness; By-production

JEL Classification Codes: C61; Q52

Nonparametric Measures of Efficiency in the Presence of Undesirable Outputs:

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As the threat of global warming leading to climate change seems to be increasingly real, scientists and policy makers all across the world are feeling the urgency of enacting appropriate regulations to contain environmental pollution by setting targets for reduction in the emission of greenhouse gases. In a recent study, Covert et al (2016) examine the fundamentals of global supply and demand for energy, and find that the current combination of markets and policies does not seem likely to diminish greenhouse gases. They conclude that in the absence of substantial greenhouse gas policies, the US and the global economy are not likely to stop relying on fossil fuels as the primary source of energy. Devising appropriate incentives for promoting sustainable development has become a center piece of economic policy of all nations.

In production economics, the concepts of technical efficiency and productivity have been redefined to take explicit account of the extent of environmental damage entailed side by side with the quantity of output produced from a given input bundle. When the production process of transforming inputs into outputs results in some unwanted consequences (generically described as *bad* or *undesirable* outputs) on the side, some of the standard assumptions routinely made about the underlying production technology will be violated. *This is particularly true for the free disposability assumption*. Under ordinary circumstances, it is not unreasonable to assume that one can always produce less output from the same input bundle. This simply allows the possibility of waste. But if one or more of the outputs is undesirable, why would not the firm produce less (or for that matter none) of it without altering its bundles of desirable or *good* outputs and inputs? The obvious answer to this is that it is not possible to produce less of (or to totally eliminate) the undesirable outputs while at the same time keeping the levels of the desired outputs and the inputs unaltered. This, in its turn, leads to the concepts of *weak disposability* and *null jointness*.[1] An implication of weak disposability is that the bad output can be reduced only if the good output is also reduced and/or more input is used. Similarly, two outputs are null joint if the quantity of one output can be reduced to zero only if the quantity of the other output is also reduced to zero.

In the relevant literature, a bad output has been accommodated in the production technology mainly in one of three different ways. In the first it is treated like an input, the justification being

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[1] Formal definitions of these two concepts are provided below in the methodology section.

that the good output is observed to be positively correlated with the bad output. A higher output on the farm goes hand in hand with more labor, machinery, chemical fertilizers as well as with more groundwater contamination. Notable examples of this interpretation are Baumol and Oates (1988) or Cropper and Oates (1992). While it is true, indeed, that in many cases the bad output is *observationally* indistinguishable from a regular input, the two are *conceptually* quite different. There are three important characteristics of an input. First, an input exists even before the production process starts. Second, an input is depleted in stock as production is carried out. Third, an input is subject to some processing by the producer. Now consider smoke emission from a power plant for an example. There is no smoke before the power plant started generating electricity. Second, there is more smoke, not less, in the air after more power is produced. Finally, once the smoke is emitted into the atmosphere, it is not subject to any further processing by the power generating plant. In fact, the unintended outcome is an *output* (even though undesirable) and not an *input*.

The second approach recognizes the undesirable output as an unintended outcome of the process of producing the desired output. In this sense, the good and the bad are *joint products*. Because production of the good output will inevitably generate some bad output on the side, reduction in the quantity of the bad output produced is not possible if the good output is not reduced as well. This jointness, when pushed to the extreme, implies that the bad output can be totally eliminated only if no good output is produced at all. This is the *null jointness* property mentioned above. Because the bad output cannot be reduced unilaterally, it does not satisfy *free disposability*. However, it is *weakly disposable* jointly with the good output. Introduced by Färe and Grosskopf[2], this approach has gained wide popularity in the literature and has effectively become the accepted analytical framework in environmental economics. An example of this kind of jointness in livestock production is between beef or milk (the good outputs) and the methane gas emitted from cow dung (the bad output).

In the third approach proposed by Førsund (2009) and Murty, Russell, and Levkoff (MRL) (2012) the bad output arises as a *by-product* of the process of producing the good output. In this sense, environmental pollution in the form of smoke emission is a 'collateral damage' rather than a joint product of power generation. Even though both the *joint production* and *by-production* interpretations lead to a positive correlation between the good and the bad output, the difference between the two is fundamental. Typically, the undesirable output can be traced back to a specific input (or group of inputs) which also contribute to the production of the good

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[2] See, for example, Färe et al (1989), Färe et al (1993), and Färe et al (2005).

output. In a power plant, all inputs – labor, capital, fuel, and other materials – are used for generating electricity. However, fuel is the only one of these inputs that produces smoke and leads to CO_2 emission.[3] Thus, if the inputs are categorized as *neutral* and *polluting,* fuel is the only polluting input and the bad output can be reduced by reducing the quantity of fuel. In all likelihood, a reduction in the quantity of fuel will also result in a reduction of the power output as well. But to the extent there are input substitution possibilities, it would be possible to maintain the level of the good output using more of the neutral inputs in place of fuel. For a different example, consider the use of chemical sprays for weeding in agriculture which results in groundwater contamination. One may use labor instead of chemicals for weeding, thereby reducing (or even eliminating) water pollution without reducing the output. In such cases, the bad output is weakly disposable not with the good output but with the polluting input. MRL also considered the possibility of reducing the bad output through additional abatement activities without reducing the use of the polluting input but diverting part of the neutral inputs from production of the good output to abatement of the bad output generated. For example, a paper mill may set up a water treatment plant for purification of waste water before it is discharged into the river. Such diversion of resources away from paper production will, naturally, reduce the good output (paper) while reducing the amount of the bad output (toxic waste) remaining after abatement.

In the present paper, like MRL we adopt the by-production interpretation of bad output generation. Also, we conceptualize the good output as technologically separable from the bad output. Moreover, we set up an integrated DEA optimization problem over the intersection of these two sub-technologies to measure the efficiency of a firm that produces a bad output alongside the good output. In an empirical illustration of our methodology, we use country level data for an unbalanced panel of 64 countries over the years 1986 through 2011 where per capita GDP is the good and per capita CO_2 emission is the bad output. Labor and capital are the neutral inputs while fuel is the polluting input. We compute three different measures of efficiency, respectively, from a Directional Distance Function (DDF), a good-expanding model, and a bad-contracting model. In each of these cases, however, weak disposability and null jointness is assumed between the bad output and the polluting input rather than the good and bad outputs. We then utilize our DEA results to compute opportunity costs of a targeted reduction in CO_2

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[3] In fact, approximately 65 percent of the global greenhouse gas emissions result from the combustion of fossil fuel (Covert et al, 2016).

emission in terms of required dollar amounts of reduction in per capita GDP for the individual countries in selected years.

To summarize, the main contributions of the present paper are the following:

- we explicitly model the bad output as a by-product of using the polluting input;
- we differentiate between weak disposability and costly disposability;
- we formulate a single unified optimization problem to measure the potential for simultaneously expanding the good output and reducing the bad output taking explicit account of its jointness with the polluting input;
- we show how one can measure the opportunity cost of reducing the bad output by a targeted amount in terms of the good output that must be foregone;
- we illustrate the methodology by providing for selected countries estimates of the minimum amount of per capita GDP that would have to be sacrificed if per capita CO_2 emission is to be reduced by 5%.

The rest of the paper unfolds as follows. In section 2 we provide the conceptual background as well as a brief overview of the nonparametric methodology. Section 3 formulates the alternative DEA models under different assumptions for measurement of efficiency and opportunity costs of reducing the bad output. Section 4 reports the empirical findings from the alternative DEA and reports the opportunity costs of different CO_2 reduction targets for individual countries. Section 5 is the conclusion.

2. The Nonparametric Methodology

2.1 The Technology

Consider an industry transforming bundles of n inputs into m outputs. An input vector $x \in R_+^n$ combined with an output vector $y \in R_+^m$ constitutes a feasible production plan if y can be produced from x. The production technology of an industry can be defined by the production possibility set

$$T = \{(x, y): x \text{ can produce } y\}. \quad (1)$$

An alternative definition of T in terms of the production correspondence $F(x, y) = \alpha$ is

$$T = \{(x, y) : \mathrm{F}(x, y) = \alpha \le 0\}. \qquad (2)$$

The technical efficiency of a specific input-output bundle (x^0, y^0) can be measured in many different ways. The most popular among them are:

(a) output-oriented efficiency: $\tau_y(x^0, y^0) = \frac{1}{\varphi^*}$ where $\varphi^* = \max \varphi : (x^0, \varphi y^0) \in T$;

(b) input-oriented efficiency : $\tau_x(x^0, y^0) = \theta^*$ where $\theta^* = \min \theta : (\theta x^0, y^0) \in T$; and

(c) directional (in)efficiency: $\beta^* = \max \beta : \left(x^0, y^0\right) + \beta(g^x, g^y) \in T$, where (g^x, g^y) is a direction prespecified by the analyst. For $g^x = -x^0$ and $g^y = y^0$, one gets

$$\beta^* = \max \beta : \left((1-\beta)x^0, (1+\beta)y^0\right) \in T.$$

Output-oriented efficiency considers the maximum proportional expansion of the output vector while input-oriented efficiency considers the maximum proportional contraction of the input vector. The directional (in)efficiency measures the maximum proportion by which all outputs can be scaled upwards while, at the same time, all inputs can be scaled downwards.[4]

In parametric analysis, one specifies some explicit functional form of a production function $F(x, y) = f(x) - y = 0 \Leftrightarrow y = f(x)$ in the single output case and the Distance Function $F(x, y) = D(x, y) - 1 = 0 \Leftrightarrow D(x, y) = 1$ in the multiple output case and estimates the parameters using Stochastic Frontier Analysis. In nonparametric Data Envelopment Analysis (DEA), one leaves the functional form unspecified and, instead, only makes the following (fairly general) assumptions about the technology in order to the estimate the production possibility set as the free disposal convex hull of the observed input-output vectors $(x^j, y^j)(j = 1, 2, ..., N)$:

(A1) Every observed input-output bundle is feasible by default. That is,

$$(x^j, y^j) \in T \, (j = 1, 2, ..., N);$$

(A2) Inputs are freely disposable. Thus,

$$(x, y) \in T \wedge \tilde{x} \geq x \Rightarrow (\tilde{x}, y) \in T \;;$$

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[4] Note that the output- and input-oriented efficiencies can both be obtained as special cases of the directional model – by setting $g^x = 0$ to get $\varphi = 1 + \beta$ for the output-oriented model or by setting $g^y = 0$ to get $\theta = 1 - \beta$ for the input-oriented model.

(A3) Outputs are freely disposable. Thus,

$$(x, y) \in T \wedge \tilde{y} \leq y \Rightarrow (x, \tilde{y}) \in T \text{ ; and}$$

(A4) The production possibility set is convex. Thus,

$$(x^1, y^1) \in T \wedge (x^2, y^2) \in T \wedge \lambda \in (0,1) \Rightarrow \left(\lambda(x^1, y^1) + (1-\lambda)(x^2, y^2) \right) \in T.$$

Based on the assumptions (A1-A4), an empirical estimate of the production possibility set is

$$S = \left\{ (x, y) : x \geq \sum_{j=1}^{N} \lambda_j x^j ; y \leq \sum_{j=1}^{N} \lambda_j y^j ; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0 (j = 1, 2, ..., N) \right\}. \qquad (3)$$

Note that *S* is the smallest set satisfying these assumptions.

Assumptions (A1) - (A4) imply variable returns scale (VRS) by default. If one additionally makes the constant returns to scale (CRS) assumption

(A5) $\qquad (x, y) \in T \wedge k \geq 0 \Rightarrow (kx, ky) \in T$,

one can get the CRS production possibility set

$$S^C = \left\{ (x, y) : x \geq \sum_{j=1}^{N} \lambda_j x^j ; y \leq \sum_{j=1}^{N} \lambda_j y^j ; \lambda_j \geq 0 (j = 1, 2, ..., N) \right\}. \qquad (4)$$

2.2 Bad Outputs

This standard conceptualization of the production technology is hardly appropriate for an industry with bad output specifically because the assumption of free disposability of the undesirable output is in conflict with rational behavior of the decision making agent. After all, if one could produce a lower quantity of the bad output without altering the quantities of the inputs and of the good output, it is not rational for the firm to produce *any* amount of the output that it considers bad.

For technologies involving the production of bad outputs, the free (or strong) disposability assumption is generally modified in one of two alternative ways. The most widely accepted approach is to assume that the bad output is weakly disposable and is null joint with one or more good outputs. At present a Directional Distance Function (due to Chambers, Chung, and Färe (1996)) assuming weak disposability of the bad output has become the standard analytical format for measuring efficiency in the presence of bad outputs. The other approach is to assume costly (rather than free) disposability of the bad output. This was adopted in MRL and is yet to find broader acceptance in the literature. Even though both assumptions would imply the observed positive correlation between the good and the bad outputs, they differ drastically in the underlying conceptualization of the production technology.

2.3 Joint Production

For simplicity, consider an industry producing one good output (*g*) along with a single bad output (*b*) using two inputs – x_1 and x_2. Under the assumption of free disposability of outputs,

$$(x_1^0, x_2^0, g^0, b^0) \in T \wedge (g^1, b^1) \leq (g^0, b^0) \Rightarrow (x_1^0, x_2^0, g^1, b^1) \in T. \qquad (5)$$

But under weak disposability

$$(x_1^0, x_2^0, g^0, b^0) \in T \wedge (g^1, b^1) = (\alpha g^0, \alpha b^0); \alpha \in (0,1) \Rightarrow (x_1^0, x_2^0, g^1, b^1) \in T. \qquad (6)$$

Thus, an implication of weak disposability is that although one cannot unilaterally reduce the bad output without any change in the quantities of the good output and/or any of the inputs, it is possible to reduce the quantity of the bad output simultaneously with the good output. In many cases, while the bad output is treated as only weakly disposable, the good output is considered to be freely disposable, Hence, in (6) above, if $g^1 \leq \alpha g^0, b^1 = \alpha b^0; \alpha \in (0,1),$ then $(x_1^0, x_2^0, \mathrm{g}^0, b^0) \in T \Rightarrow (x_1^0, x_2^0, \mathrm{g}^1, b^1) \in T.$ In fact, Färe , Grosskopf, and Lovell (1994) first construct the production possibility set with weak disposability of the bad output along with free disposability of inputs and the good output as

$$S_{JP}^{WD} = \left\{ \begin{array}{l} (x_1, x_2; g, b): x_i \geq \sum_{j=1}^{N} \lambda_j x_{ij}; (i = 1, 2); \\ g = \alpha \sum_{j=1}^{N} \lambda_j g_j;\ b = \alpha \sum_{j=1}^{N} \lambda_j b_j; \\ 0 \leq \alpha \leq 1; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, ..., N). \end{array} \right\} \qquad (7)$$

Next setting $\alpha = 1$ and assuming free disposability of the good output, they specify

$$S_{JP}^{WD} = \left\{ \begin{array}{l} (x_1, x_2; g, b): x_i \geq \sum_{j=1}^{N} \lambda_j x_{ij}; (i = 1, 2); \\ g \leq \sum_{j=1}^{N} \lambda_j g_j; b = \sum_{j=1}^{N} \lambda_j b_j; \\ \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, ..., N). \end{array} \right\} \quad (8)$$

However, the equality constraint in respect of the bad output in (8) above actually makes it non-disposable rather than weakly disposable.

2.4 Implications of Disposability

At this point, one needs a clear understanding of what free disposal of any output (or input) means in real life. Färe et al (1994, page 38) describe weak disposability as the ability of the producer to dispose of an unwanted commodity at a positive private cost. Førsund (2009, fn 11) argues that free disposability does not imply costless removal of output already produced. However, while one can 'abstain from' producing something, one cannot 'dispose of' something that has not been actually produced. Førsund interprets free disposability in terms of monotonicity of the transformation function. Specifically, when all outputs and inputs are freely disposable,

$\frac{\partial F}{\partial x_i} \leq 0$ for any input x_i (i) and

$\frac{\partial F}{\partial y_j} \geq 0$ for any output y_j. (ii)

These signs of the partial derivatives of the transformation function ensure that along the graph of the technology $G = \{(x, y) : F(x, y) = 0\}$, $\frac{dx_i}{dx_j} \leq 0, \frac{dy_r}{dy_s} \leq 0,$ and $\frac{dy_r}{dx_j} \geq 0.$ While this argument is quite reasonable, the question is: what happens when one moves to a point off the frontier?

First consider a conventional 2-output 2-input case where there is no bad output. Suppose $\text{F}(x_1^0, x_2^0; y_1^0, y_2^0) = 0$ and, hence, the bundle $(x_1^0, x_2^0; y_1^0, y_2^0)$ is on the frontier. Next consider a smaller quantity of output 1 $(y_1^1 < y_1^0)$. Now, by free disposability of output 1, $\text{F}(x_1^0, x_2^0; y_1^1, y_2^0) < 0$ and output bundle (y_1^1, y_2^0) can be produced from the input bundle (x_1^0, x_2^0). But what exactly is going on with the quantity of output 1? If the inputs used are unchanged and the other output is also unchanged, then there are three possibilities: (a) due to less intensive utilization of the same bundle of inputs less of output 1 is actually produced; (b) the same quantity of output 1 is produced but is allowed to go waste; and (c) some of the inputs are left unused. The first case is one of inefficiency. An example of case (b) would be one where some of the crop grown in a farm is not harvested and is left to rot on the field. Case (c) could arise in one of two ways. It may be that some of the input is wasted in the production process and is not available for use in the future even though it is not fully utilized for production now. Again, this is inefficiency and is not distinguishable from case (a). Alternatively, it could be that some of the input is simply left in the warehouse (and is available for future use) in which case a reduction in output 1 is accompanied by a reduction in the input bundle. Førsund prefers to rule out the case (b).

Now return to the case where the two outputs are *g* (good) and *b* (bad). Under the weak disposability assumption, *b* can be reduced only simultaneously (and proportionately) with *g*. In this case, disposal of the bad output is costly in the sense that there has to be a reduction in the good output in order to accomplish a reduction in the bad output. This is quite consistent with the interpretation that the good and bad outputs are joint products. Note that simultaneous reduction in *g* and *b* without any change in the inputs can happen either because of inefficiency (underutilization of inputs) or diversion of resources from the production of good output to abatement of the bad output that has already been generated. The latter is the abatement option considered by MRL.

2.5 Weak or Costly Disposability?

There is also a subtle difference between weak disposability and costly disposability. Weak disposability of any commodity (output or input) has to be in conjunction with something else. An output or an input cannot be weakly disposable by itself. That is not necessarily true for costly disposability. For example, in the case of water quality treatment, one can reduce the bad output (water pollution) simply by devoting resources for pollution abatement – an option that is costly for the producer. Another point to note is that even when the bad output is weakly disposable together with the good output, one can still assume free disposability of the good output (in a restricted sense). We have seen that in the case of weak disposability

$(x_1^0, x_2^0, \mathrm{g}^0, b^0) \wedge 0 \le \alpha < 1 \in T \Rightarrow (x_1^0, x_2^0, \alpha\,\mathrm{g}^0, \alpha b^0) \in T.$ But when, additionally, the good output is assumed to be freely disposable,

$$(x_1^0, x_2^0, \mathrm{g}^0, b^0) \wedge 0 \le \delta < \alpha < 1 \in T \Rightarrow (x_1^0, x_2^0, \delta\,\mathrm{g}^0, \alpha b^0) \in T. \qquad (9)$$

In this case, a 10% reduction in smoke emission must be accompanied by a 10% reduction in power generation, due to weak disposability of the bad output. However, production inefficiency may lead to more than 10% reduction in power. By setting α equal to unity, one can trivially obtain free disposability of the good output.

2.6 By-production

We now focus on weak disposability of the bad output and the polluting input. In power generation, fossil fuel is the source of carbon emission and a reduction in CO_2 emission (in the absence of pollution abatement) requires lowering the use of fossil fuels. If there is no inter-fuel substitution possibility, burning less fuel will result in a lower quantity of the good output (power). However, power and smoke *are not joint products*. Rather, the bad output (smoke) is an unavoidable by-product of the use of fossil fuel for power generation. Following MRL, one can conceive of two separate sub-technology sets:

$$T^g = \left\{(x_1, x_2; g) : F^g(x_1, x_2; g) \le 0; \tfrac{\partial F^g}{\partial x_i} < 0\,(i = 1, 2); \tfrac{\partial F^g}{\partial g} > 0\right\} \quad (10)$$

and

$$T^b = \left\{(x_2; \mathrm{b}) : F^b(x_2; \mathrm{b}) \le 0; \tfrac{\partial F^b(kx_2, kb)}{\partial k} < 0, \tfrac{\partial F^b}{\partial x_2} < 0\right\}. \quad (11)$$

The sub-technology set in (11) shows that the bad output will increase or decrease only in conjunction with the polluting input. But one may also simply waste the polluting input without causing any more of the bad output. As visualized in (11) the only way to reduce the bad output is to lower the use of the polluting input.

If, on the other hand, we allow remedial treatment of the bad output already produced in the act of producing the good output, the sub-technology in (11) would be revised as

$$T^b = \left\{(x_1, x_2; \mathrm{b}) : F^b(\mathrm{x}_1, x_2; \mathrm{b}) \le 0; \tfrac{\partial F^b(kx_2, kb)}{\partial k} < 0, \tfrac{\partial F^b}{\partial x_1} < 0, \tfrac{\partial F^b}{\partial x_2} < 0\right\}. \quad (11a)$$

The overall production possibility set corresponding to (10) and (11) is

$$T = \left\{(x_1, x_2; g, b) : (x_1, x_2; g) \in T^g \wedge (x_2; b) \in T^b\right\}. \quad (12)$$

The corresponding nonparametric construction of the production possibility set will be

$$S_{BP}^{WD} = \left\{ \begin{array}{l} (x_1, x_2; g, b) : x_1 \geq \sum_{j=1}^{N} \lambda_j x_{1j}; x_2 \geq \alpha \sum_{j=1}^{N} \lambda_j x_{2j}; \\ g \leq \sum_{j=1}^{N} \lambda_j g_j; b = \alpha \sum_{j=1}^{N} \lambda_j b_j; \\ 0 \leq \alpha \leq 1; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, ..., N). \end{array} \right\} \qquad (13)$$

3. DEA Models

3.1 Alternative Measures of Performance

In the present paper we propose three different efficiency measures: good output-oriented efficiency, bad output-oriented efficiency, and the output-oriented directional inefficiency. For our analysis we adopt the by-production interpretation under weak disposability of the bad output and the polluting input. Also, using the directional projection as the reference point on the frontier, the good and the bad oriented projections provide discrete measures of the tradeoff between the good and the bad output in the upward and in the downward directions.

A simple diagram will be useful at this point. Figure 1 shows the Output Set

$$P(x_1^0, x_2^0) = \left\{ (g, b) : (x_1^0, x_2^0; g, b) \in T_{BP}^{WD} \right\}$$

for a fixed input bundle (x_1^0, x_2^0). The points *P₁* through *P₅* show the observed quantities of the good and the bad output produced by five firms. The piece-wise connected line $OP_1P_2P_4P_5Q$ is the boundary of the output set. We are interested in measuring the efficiency of Firm 3 producing the output bundle (g_0, b_0) shown by the point P_3. Point A on the P_2P_4 segment of the frontier is the good-output-oriented projection with outputs $(g^A = \varphi g_0; b^A = b_0)$. Similarly, its bad-output-oriented projection on to the frontier is the point B on the OP_1 segment with outputs $(g^B = g_0; b^B = \theta b_0)$. Finally, the point C with outputs $(g^C = (1+\beta) g_0; b^C = (1-\beta) b_0)$ on the P_1P_2 segment is its directional projection onto the frontier in the direction of the point $F\,(g = g_0; b = -b_0)$.

Correspondingly, the alternative efficiency indicators of Frm 3 (shown by the point $P3$) are the good-oriented efficiency:

$$\tau_g(g_0, b_0 \mid x_1^0, x_2^0) = \tfrac{1}{\varphi} = \frac{b_0 P_3}{b_0 A};$$

the bad-oriented efficiency:

$$\tau_b(g_0, b_0 \mid x_1^0, x_2^0) = \theta = \frac{g_0 B}{O b_0}; \text{ and}$$

the output directional inefficiency:

$$\vec{\mathrm{D}}(g_0, b_0 \mid x_1^0, x_2^0) = \beta = \frac{CP_3}{OF}.$$

The relevant DEA optimization problem for measuring good-output-oriented technical efficiency under the weak disposability and by-production assumption is

$$\begin{aligned}
&\max\ \varphi \\
&s.t. \\
&\sum_{j=1}^{N} \lambda_j g_j \geq \varphi g_0; \sum_{j=1}^{N} \lambda_j x_{1j} \leq x_1^0; \\
&\sum_{j=1}^{N} \lambda_j b_j = \alpha b_0; \sum_{j=1}^{N} \lambda_j x_{2j} \leq \alpha x_2^0; \\
&0 \leq \alpha \leq 1; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, \ldots, N).
\end{aligned} \tag{14}$$

For the bad-output-oriented projection, we need to solve the DEA problem

$$\begin{aligned}
&\min \theta \\
&s.t. \\
&\sum_{j=1}^{N} \lambda_j g_j \geq g_0; \sum_{j=1}^{N} \lambda_j x_{1j} \leq x_1^0; \\
&\sum_{j=1}^{N} \lambda_j b_j = \theta b_0; \sum_{j=1}^{N} \lambda_j x_{2j} \leq \theta x_2^0; \\
&\sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, \ldots, N).
\end{aligned} \tag{15}$$

Note that in this problem the constraint $\theta \leq 1$ is redundant.

Finally for the directional inefficiency measurement, the DEA problem is

$$
\begin{aligned}
&\max\ \beta \\
&s.t. \\
&\sum_{j=1}^{N} \lambda_j g_j \geq (1+\beta) g_0; \sum_{j=1}^{N} \lambda_j x_{1j} \leq x_1^0; \\
&\sum_{j=1}^{N} \lambda_j b_j = (1-\beta) b_0; \sum_{j=1}^{N} \lambda_j x_{2j} \leq (1-\beta) x_2^0; \\
&\sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, \ldots, N).
\end{aligned} \tag{16}
$$

Again, in this problem β will naturally be bounded above by unity because otherwise the benchmark level of the bad output (and the polluting input) will become negative.[5]

3.2 Opportunity Costs

While the summary measures of efficiency (or inefficiency) are somewhat useful for comparing performance across units, a more relevant question centers around the opportunity cost of reducing the bad output in terms of the quantity of the good output foregone. It should be emphasized, however, that for any unit located in the interior of the output set of its input bundle, there will remain room for unilateral increase in the good output or reduction in bad output to some extent. For such units, initially there is no opportunity cost. A tradeoff between the good and the bad output becomes meaningful only after any technical inefficiency has been removed and the observed output bundle has been projected onto the frontier. But that, by itself, does not remove all ambiguities. First, as we have seen above, there are alternative ways to project an inefficient point onto the frontier. Second, given the nonlinear nature of the frontier, change in the good output that accompanies a change in the bad output as one moves from one point to another on the frontier depends on both the direction and the size of the change.[6] For an inefficient point, the directional projection is in a sense the more logical choice for a

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[5] In the joint production approach the constraint on input 2 will be $\sum_{j=1}^{N} \lambda_j x_{2j} \leq x_2^0$

[6] Several authors have used the dual variables of the DEA LP problem to compute shadow prices or marginal rates of transformation between the good and the bad outputs along the frontier. However, often these multipliers are not unique. Besides they are usually extremely unstable and are not useful for measuring opportunity costs of discrete changes. For this approach, see Färe and Grosskopf (1998), Lee, park, and Kim (2001), Färe et al (2005), Salnykov and Zelenyuk (2005), and Ray and Mukherjee (2007).

reference point on the frontier because it lies in between the bad-oriented and the good-oriented projections. Also, comparison of the output bundles at this point and at the bad-oriented projection yields a measure of the tradeoff, $\left(\frac{\Delta g}{\Delta b_-}\right)$, when the bad output is reduced. On the other hand, comparing the reference bundle with the good-oriented projection provides a measure of the tradeoff, $\left(\frac{\Delta g}{\Delta b_+}\right)$, as the good output is increased.[7]

In the example shown in Figure 1, the movement from the directional reference point to the bad-oriented projection is from C to B. In order to achieve a reduction in the bad output by BE, the reduction required in the good output is CE. At the point C the good output is $(1+\beta)g_0$ and at B it is just g_0. Hence, the reduction in the good output is $CE = \Delta g = (1+\beta)g_0 - g_0 = \beta g_{0.}$ On the other hand, the quantity of the bad output at B is θb_0 whereas at C it is $(1-\beta)b_0$. Thus, the change in the bad output is $BE = \Delta b = (1-\beta)b_0 - \theta b_0$. Hence, the downward tradeoff is

$$\frac{\Delta g}{\Delta b_-} = \frac{\beta g_0}{(1-\beta-\theta)b_0}. \qquad (17)$$

For the upward tradeoff, we compare the points C and A. This time, the increase in the good output is $\Delta g = AD = \varphi g_0 - (1+\beta)g_0 = (\varphi - 1 - \beta)g_0$. The corresponding increase in the bad output is $\Delta b = CD = b_0 - (1-\beta)\mathrm{b}_0 = \beta b_0$. The upward tradeoff is

$$\frac{\Delta g}{\Delta b_+} = \frac{(\varphi - 1 - \beta)g_0}{\beta b_0}. \qquad (18)$$

In order to derive the downward and upward elasticities we need to scale the changes- Δg and Δb - by the quantities of the good and the bad output at the reference point $(g^*, b^*) = \left((1+\beta)g_0, (1-\beta)\mathrm{b}_0\right)$. The resulting downward elasticity at the reference point is

$$\varepsilon_- = \frac{\Delta g}{\Delta b_-} . \frac{b^*}{g^*} = \frac{\beta g_0}{(1-\beta-\theta)b_0} . \frac{(1-\beta)b_0}{(1+\beta)g_0} = \frac{\beta(1-\beta)}{(1-\beta-\theta)(1+\beta)}. \qquad (19)$$

---

[7] An increase in the good output even with a corresponding increase in the bad output may not be feasible without any increase in any input.

Similarly, the upward elasticity is

$$\varepsilon_{+} = \frac{\Delta g}{\Delta b_{+}} . \frac{b^{*}}{g^{*}} = \frac{(\varphi - 1 - \beta)(1-\beta)}{\beta(1+\beta)}. \qquad (20)$$

While the tradeoffs shown in (17-18) or the elasticities in (19-20) are useful measures of the opportunity cost of reducing the bad output for inefficient units, they cannot be used if the unit is already on the frontier. For such units one needs to consider the change in the quantity of the good output corresponding to a reduction (or an increase) in the bad output by a prespecified proportion.

Suppose that some unit with input-outputs $(x_1^0, x_2^0; g_0, b_0)$ is efficient. Now suppose that we are interested in the opportunity cost of reducing the bad output by $\Delta b_{-} = \gamma b_0$ where γ is a proportion chosen by the analyst. Because of weak disposability of the bad output jointly with the polluting input, x_2 also would have to be reduced by $\Delta x_2 = \gamma x_2^0$. We want to find the corresponding change in the good output Δg_{-} such that $(x_1^0, x_2^0 - \Delta x_2; g_0 - \Delta \mathrm{g}_{-}, b_0 - \Delta \mathrm{b}_{-})$ is also efficient. For this we solve the LP problem

$$\begin{aligned} &\varphi_{-} = \max\ \varphi \\ &s.t. \\ &\sum_{j=1}^{N} \lambda_j g_j \geq \varphi g_0; \sum_{j=1}^{N} \lambda_j x_{1j} \leq x_1^0; \\ &\sum_{j=1}^{N} \lambda_j b_j = (1-\gamma) b_0; \sum_{j=1}^{N} \lambda_j x_{2j} \leq (1-\gamma) x_2^0; \\ &\sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, ..., N). \end{aligned} \qquad (21)$$

Note that in this problem, γ is a parameter. We can now obtain $\Delta g_{-} = (1-\varphi_{-}) g_0$ and the downward tradeoff is

$$\frac{\Delta g_{-}}{\Delta b_{-}} = \frac{(1-\varphi_{-}) g_0}{\gamma b_0} \qquad (22)$$

and the downward elasticity is

$$\varepsilon_{-} = \frac{\Delta g_{-}}{\Delta b_{-}} = \frac{(1-\varphi_{-})}{\gamma}. \qquad (23)$$

For the downward tradeoff we considered the decline in the maximum producible amount of the good output as the bad output is reduced (and the polluting input that contributes the good output is also reduced). For the upward tradeoff, we ask the reverse question: by how much will the minimum level of the bad output increase when we increase the good output by $\Delta g_{+} = \gamma g_0$? This time we solve the problem

$$\begin{aligned}
&\theta_{+} = \min \theta \\
&s.t. \\
&\sum_{j=1}^{N} \lambda_j g_j \geq (1+\gamma) g_0; \sum_{j=1}^{N} \lambda_j x_{1j} \leq x_1^0; \qquad (24) \\
&\sum_{j=1}^{N} \lambda_j b_j = \theta b_0; \sum_{j=1}^{N} \lambda_j x_{2j} \leq \theta x_2^0; \\
&\sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; (j = 1, 2, \ldots, N).
\end{aligned}$$

As before, γ is a parameter. Also, because the target level of the good output is higher than g_0, $\theta_{+} > 1$. The minimum increase in bad output is $\Delta b_{+} = (\theta_{+} - 1) b_0$. The upward tradeoff is

$$\frac{\Delta g_{+}}{\Delta b_{+}} = \frac{\gamma g_0}{(\theta_{+} - 1) b_0} \qquad (25)$$

And the upward elasticity is

$$\varepsilon_{+} = \frac{\gamma}{(\theta_{+} - 1)}. \qquad (26)$$

For consistency, one should first project all units to the directional efficient point. For units that are already efficient, β equals 0 and there is no change. For inefficient points, the tradeoffs in (22) and (25) become

$$\frac{\Delta g_{-}}{\Delta b_{-}} = \frac{(1-\varphi_{-})(1+\beta) g_0}{\gamma (1-\beta) b_0} \qquad (27) \qquad \text{and}$$

$$\frac{\Delta g_{+}}{\Delta b_{+}} = \frac{\gamma (1+\beta) g_0}{(\theta_{+} - 1)(1-\beta) b_0} \qquad (28)$$

There is no change in the elasticity measures because the proportionate changes are defined with respect to the efficient point as the reference.

4. The Empirical Application

We conceptualize a production technology with two outputs, one good and one bad, and three inputs, of which two are neutral and one is polluting. The good output is GDP and the bad output is CO_2 emissions. Labor and Capital are the neutral inputs while Fuel is the polluting input. All outputs and inputs are measured per capita to adjust for country/population size. Per capita GDP and Capital are both measured in purchasing power adjusted constant 2005 US dollars. Per capita CO_2 emission is measured in tonnes. The labor input is essentially the proportion of the population that is employed. Fuel is measured by tonnes of oil equivalent per capita. Data for GDP, Labor, Capital, and population are obtained from Penn World Tables. Data for fuel consumption and also on Carbon emission are obtained from BP Energy Year Book (various years). Given that the data are at the country level, we assume CRS.

The by-production model (16) was solved to evaluate the Directional Distance Function (β) for each of the 64 countries in the unbalanced panel data set over the years 1986 through 2011.[8] Table 1 reports the average yearly values of (β) for different sub-periods as well as for the entire sample period (1986-2011) for selected countries. [9] Over the entire period, France had the lowest average inefficiency (0.0929) among the 10 countries reported in Table 1. A value of β equal to 0.0929 implies that on an average France can increase per capita GDP and at the same time reduce per capita CO_2 emission by 9.29%. Not surprisingly, Russia showed the most inefficiency. Finland also performed poorly followed by Brazil. Britain, China, India, and Japan were more or less comparable. The US was slightly more inefficient that France. If we compare the average over the shorter period 1990-2011 with the longer period 1986-2011, the results change only slightly. Breaking up the 3-decade interval 1990-2011 into 3 smaller sub-periods – 1990-96, 1997-2004, and 2005-2011 – tells an interesting story. China was quite efficient with a directional inefficiency measure of a little over 3% during 1990-96. But during the subsequent periods of rapid industrialization inefficiency became much worse increasing to 20% during 2005-11. For a number of countries (Finland, France, and Japan) efficiency improved during 1997-2004 but during 2005-11 returned to levels comparable to what they were during 1997-

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[8] Our sample includes 52 countries in 1986, 53 in 1987, 54 in 1988 and 1989, and 64 in the years 1990 through 2011.

[9] We report the results for 10 countries from our sample – four emerging economies Brazil, China, India, and Russia and six advanced (OECD) economies Finland, France, Germany, Japan, United Kingdom, and United States. Data for Russia was available from 1990 onwards.

2004. Brazil (like China) became more inefficient during 1997-2004 but stayed at that level during the subsequent period. UK showed considerable improvement in efficiency during 1997-2004 and even though it performed worse in the later years, inefficiency was only half of what it was in the previous sub period. India registered a surprising decline in inefficiency from 19.35% during 1990-1996 to 13.17% during 1997-2004 down all the way to a mere 2.64% during 2005-2011. This is explained by the fact it ended up with β equal to 0 during the entire period 2008-2011. Table 1 also reports the values of β for the years 2008-09, the beginning of the great recession. Except for India and (to a certain extent UK) all countries showed significant inefficiency.

Table 2 provides for illustrative purposes a comparison of the efficiency/inefficiency measures for these 10 countries (for years 2008 and 2011) obtained from the good-output oriented model (14), the bad output oriented model (15), the by-production model (16), and a Färe-Grosskopf type joint-product model (a modified version of 16), where weak disposability holds between good and bad outputs rather than between the good output and the polluting input. In this table, good output oriented inefficiency can be measured by $(\varphi-1)$, the bad oriented inefficiency as $(1-\theta)$ and can be compared with the directional inefficiencies reported as β_{BP} and β_{JP}, for the downward tradeoff - by-production and joint production technologies. In 2008, for Germany the good and bad oriented inefficiencies are quite close $(\varphi-1)=0.2965$ and $(1-\theta)=0.2663$. Similarly, for Japan $(\varphi-1)=0.4035$ and $(1-\theta)=0.3246$. For the remaining countries, the good and bad oriented inefficiencies are quite different. For UK, the good output can be expanded only 2.83% but the bad can be reduced by nearly 20%. Similarly, for the US, per capita CO_2 emission can be cut by more than 50% but the per capita GDP can be increased by only about 16%. Of course, India was on the frontier and there was no identifiable inefficiency by any measure. As for directional inefficiencies (β_{BP} and β_{JP}) the two measures are quite close to each other. Also, given that unlike the by-production model the joint production model does not put any special constraints on the polluting input (energy), it is not surprising that β_{JP} is generally somewhat bigger than β_{BP}. In this context the large difference between the two for China in both the years is quite informative. Notice that in the by-production model β is the rate of proportional increase in the good output side by side with reduction in the bad output and in the use of energy at the same rate. There is no such constraint on the use of energy in the joint production model. In other words, it is assumed that the bad output can be reduced by simply cutting down the production of good output without explicitly requiring that use of energy, the polluting input, also must be reduced to make it possible. This accounts for the large difference between β_{BP} and β_{JP} in the case of China where there is intensive use of fuels for production.

Finally, in Table 3 we show the downward tradeoff - reduction in per capita GDP- as an opportunity cost of a 5% reduction in CO_2 emission per capita. As already stated, any opportunity cost should be measured along the efficient frontier of the output set. Therefore, all inefficient observations (g_0, b_0) were first projected on to the frontier at the directional efficient points $(g^* = (1+\beta)g_0; b^* = (1-\beta)b_0)$. Thereafter the LP problem in (21) was solved for $\gamma = 0.05$. The results from 2008 show that a 5% reduction in emission would cause a 5% reduction in per capita GDP in China and slightly less than 5% reduction in France, Germany, and India. At the other end, for Russia, UK, and the US, the opportunity cost would less than 0.05% of per capita GDP foregone.

Given that the countries differ considerably both in per capita GDP (ranging from a high of \$48647.18 for US to a low of \$3111.217 for India) and per capita CO_2 emission (17.95 tonnes in the US and 1.21 tonne in India) in 2008, it is more informative to look at the quantities (rather than percentages) of change. For both Germany and Japan, a reduction in per capita emission by 0.45 tonne will lower per capita GDP by about \$1890. A comparable reduction in emission in UK and Russia will lower per capita GDP by less than \$150. By contrast, a much smaller reduction in emission in India (by only 0.06 tonne per capita) will lower per capita GDP by a comparable amount of \$144. At the other end, pollution reduction by nearly 0.9 tonne per capita will cost the US a reduction in per capita GDP by \$394. The results reported for 2011 are to be similarly interpreted.

Strangely enough, the result for Russia in 2011 is quite counter intuitive. In this case, it appears that a reduction in the bad output actually would *raise* rather than *lower* the good output. This is quite difficult to explain and appears to be a result of weak or costly disposability of the bad output. A value of $\varphi_- > 1$ implies that along the frontier as the bad output declines the good output increases. This is shown in Figure 2. Notice that the convex hull of the observed points A, B, C, D, E, and F is the closed area ABDE. Assuming weak disposability of the good and the bad outputs along with free disposability of the good output, one gets the output set *OBDEJ*. Now consider the points C and F both of which are interior points. Now, point F is projected in the direction OF' to the point G which is on the upward sloping segment of the frontier. Point C, on the other hand is projected in the direction OC' at the point H which is on the downward sloping segment of the frontier. For all points beyond B on the frontier, the good and bad outputs would change in opposite directions. But, how can one intuitively explain this apparent paradox? One possible explanation may lie in diminishing marginal productivity in pollution abatement. Suppose that CO_2 emissions pose a serious health hazard and affects worker productivity. In that case inputs diverted to pollution abatement may cause the productivity of the resources remaining in the production of the good output to improve significantly so that in the end, the good output increases even as the bad output is reduced.

Another point needs to be emphasized. In creating the benchmark for evaluating the performance of a unit, we have not paid any attention to the composition of the GDP of either the unit under evaluation or of the hypothetical benchmark. Consider, for example, the US or China in 2008. The β_{BP} for US implies that the benchmark unit would produce 13% more of GDP with 13% less of CO_2 emission. However, mining, manufacturing, and utilities (MMU) produced a mere 16.7% of the actual GDP in the US in 2008 whereas the share of MMU in the GDP of the benchmark unit would be 39.7%. This is a drastic change in the composition of the output and cannot be attained without significant adjustment costs. For China during the same year the actual share of MMU is 41.2% while the share in the benchmark unit would be only 20.15%. Again, this is not a realistic benchmark.

This does not, however, invalidate the benchmarking procedure that is recommended in this paper. All we need to do is to incorporate an additional constraint in the relevant DEA LP problem to ensure that that the share of MMU in the benchmark unit lies within an acceptable interval of $\pm\delta$ around the actual share of MMU thereby avoiding any drastic change in the composition of the GDP. Finally, at a plant level application, the question of a change in the output mix across firms within the same industry would not be of particular concern.

5. Conclusion

Appropriate characterization of the production technology is essential for both performance evaluation and designing effective remedial policies in situations where the production process results in undesirable outcomes side by side with the intended outputs. This paper formulates an analytical model for the case where the undesirable output is traced to some ‘offending’ input and the bad output can be reduced either by reducing the use of the particular input or by diverting other inputs to actively neutralize the bad output already created. The accompanying empirical application shows how to measure the opportunity cost of lowering the undesirable outcome in terms of the good output that has to be foregone.

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Table 1: Average Betas for Selected Countries

Country	**1986-2011**	**1986-1996**	**1990-2011**	**1990-1996**	**1997-2004**	**2005-2011**	**2008-2009**
Brazil	0.2229	0.1650	0.2360	0.1731	0.2635	0.2675	0.2044
China	0.1319	0.0482	0.1425	0.0336	0.1875	0.2000	0.1802
Finland	0.3096	0.3047	0.3225	0.3424	0.2936	0.3356	0.2484
France	0.0929	0.0984	0.0971	0.1147	0.0563	0.1262	0.1462
Germany	0.2224	0.2712	0.2099	0.2599	0.1889	0.1840	0.1632
India	0.1458	0.2320	0.1179	0.1935	0.1317	0.0264	0.0000
Japan	0.1666	0.2024	0.1505	0.1726	0.1033	0.1825	0.2012
Russia	0.5960	0.5841	0.5960	0.5841	0.7643	0.4157	0.2991
United Kingdom	0.1280	0.1997	0.1216	0.2204	0.0453	0.1100	0.0668
United States	0.1046	0.0762	0.1237	0.1198	0.0975	0.1574	0.1546

Table 2: Comparison of Results across Models

2008				
Country	φ	θ	β	β_{FG}
Brazil	1.3131	0.5523	0.1986	0.2220
China	1.4451	0.6910	0.1828	0.4881
Finland	1.3365	0.5310	0.2282	0.2582
France	1.2035	0.6406	0.1469	0.1641
Germany	1.2965	0.7337	0.1516	0.2037
India	1	1	0.0000	0.0000
Japan	1.4035	0.6754	0.1912	0.2967
Russia	1.3479	0.2952	0.2812	0.3097
United Kingdom	1.0283	0.8040	0.0257	0.0268
United States	1.1592	0.4888	0.1355	0.1412
2011				
Country	φ	θ	β	β_{FG}
Brazil	1.3605	0.4909	0.2368	0.2670
China	1.4642	0.6830	0.1884	0.5825
Finland	1.4366	0.4971	0.2968	0.2992
France	1.2147	0.6333	0.1424	0.1476
Germany	1.2083	0.6711	0.1690	0.1861
India	1	1	0.0000	0.0000
Japan	1.4609	0.6412	0.2186	0.3524
Russia	1.3379	0.2871	0.2891	0.2957
United Kingdom	1.1492	0.7048	0.1159	0.1272
United States	1.0257	0.4302	0.0251	0.0295

Table 3: Tradeoffs between per capita GDP and Pollution: $\gamma=0.05$

2008					
Country	φ_{-}	$g^{*}=(1+\beta)\,\mathrm{g}_0$	$b^{*}=(1-\beta)b_0$	$\Delta g^{*}=(1-\varphi_{-})g^{*}$	$\Delta b^{*}=(1-\gamma)\,\mathrm{b}^{*}$
Brazil	0.9807	9870.33	1.8249	190.1896	0.0912
China	0.95	7831.728	4.2286	391.5864	0.2114
Finland	0.9841	41134.98	7.8647	652.9148	0.3932
France	0.9857	34251.53	5.6328	490.7392	0.2816
Germany	0.9513	38827.69	8.7830	1891.317	0.4392
India	0.9537	3111.217	1.2125	143.9971	0.0606
Japan	0.9513	38481.53	8.9627	1872.741	0.4481
Russia	0.9933	21452.9	8.4761	142.8095	0.4238
United Kingdom	0.9953	31518.03	9.1875	148.6116	0.4594
United States	0.9919	48647.18	17.9491	394.9184	0.8975
2011					
Country	φ_{-}	$g^{*}=(1+\beta)\,\mathrm{g}_0$	$b^{*}=(1-\beta)b_0$	$\Delta g^{*}=(1-\varphi_{-})g^{*}$	$\Delta b^{*}=(1-\gamma)\,\mathrm{b}^{*}$
Brazil	0.9839	10709.05	1.8851	172.5222	0.0943
China	0.9500	9732.096	5.3158	486.6048	0.2658
Finland	0.9812	42277.77	6.7257	793.5995	0.3363
France	0.9865	34273.64	5.0446	463.976	0.2522
Germany	0.9835	40721.09	8.1141	673.5876	0.4057
India	0.9514	3755.14	1.3691	182.3421	0.0685
Japan	0.95	38833.94	8.1415	1941.697	0.4071
Russia	1.0014	21679.35	8.5052	-30.0195	0.4253
United Kingdom	0.9825	34247.96	7.1969	599.1366	0.3598
United States	0.9989	43196.01	18.6881	49.02023	0.9344

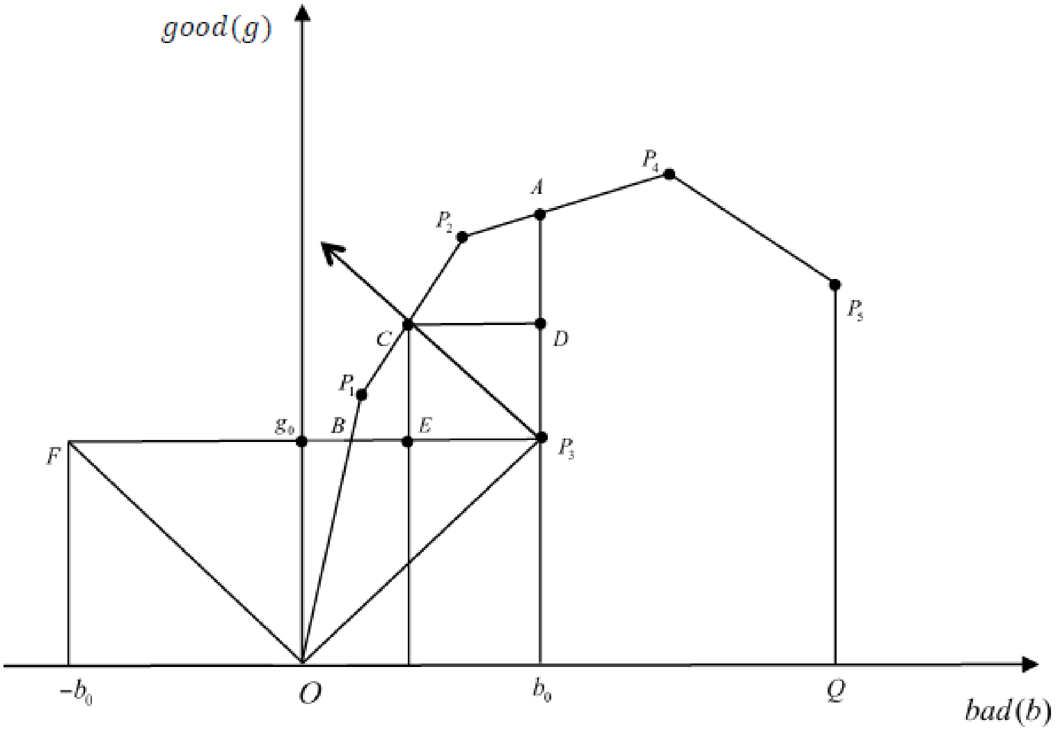

Figure 1. Efficient Directional Projection

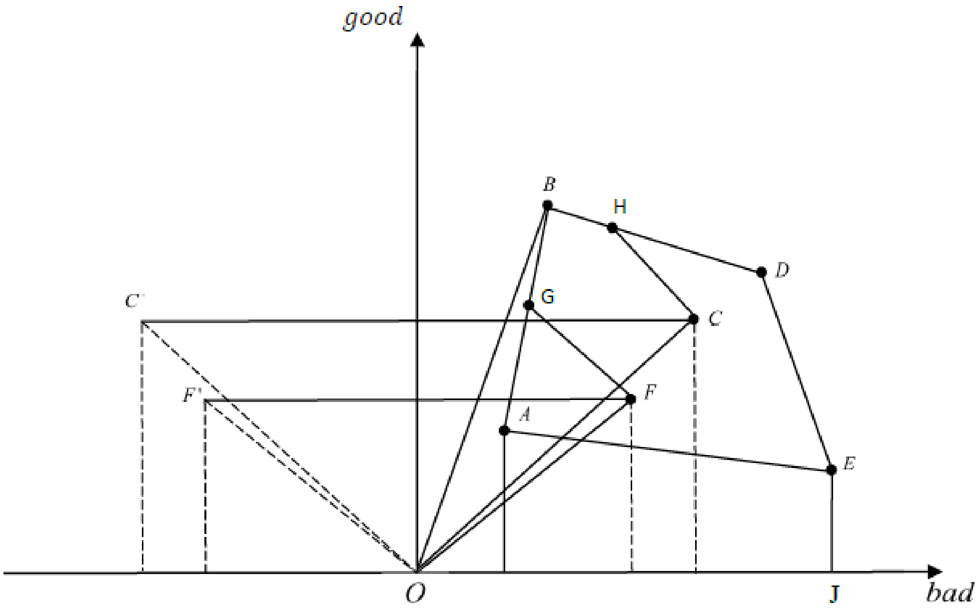

Figure 2. Directional Projections to the Frontier