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Decomposition of Profit Efficiency under Alternative Definitions of Technical Efficiency

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DECOMPOSITION OF PROFIT EFFICIENCY UNDER ALTERNATIVE DEFINITIONS OF TECHNICAL EFFICIENCY

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Abstract:

In this paper we consider alternative multiplicative decompositions of profit efficiency measured by the ratio of the actual profit of a firm to the maximum achievable profit. We use the directional distance function of Chambers, Chung, and Färe (1996) as the generic analytical method for measuring technical efficiency. Alternative choice of directions yields the Shephard input and output distance functions as well as the lesser-known McFadden's gauge function for measuring technical efficiency. We also present an endogenously determined direction based on the overall efficiency measure of Ray (2007). We show that the McFadden gauge function is the only one that yields a decomposition where both the technical and allocative components of profit efficiency lie between 0 and 1 while the technical efficiency factor is also independent of prices.

JEL Classification. D24, C61

Keywords: Directional Distance Function; Gauge Function; Nerlovian Efficiency

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1. Introduction

In neoclassical production economics the objective of a competitive firm acting as a price-taker in both product and input markets is to maximize profit by selecting an optimal input-output bundle that is also technically feasible. In fact, profit maximization by firms and utility maximization by consumers together ensure Pareto efficient allocation of resources in the model of a perfectly competitive economy. It is important, therefore, to examine if, and to what extent, a firm has failed to attain the maximum profit possible given the vectors of input and output prices and also the technology it has access to. When the output bundle is treated as a predetermined target, minimizing cost is a natural objective to pursue and cost efficiency is an appropriate criterion for the evaluation of the productive performance of a firm. In fact, in an overwhelming majority of empirical research the main focus is on measurement and decomposition of cost efficiency both in the short and in the long run. However, in the textbook analysis of producer's behavior a firm selects both inputs and output(s) simultaneously in order to maximize profit subject to the constraint that the chosen input-output bundle must be feasible. If the actual input-output bundle of a firm is in the interior of the production possibility set, it is technically inefficient and there is room to scale up the output without changing the input or to scale down the input without changing the output (or to increase output while reducing input simultaneously). In any one of these cases, the profit will increase. Such increase in profit is ascribed to eliminating technical inefficiency. But while all points on the frontier are technically efficient, they do not yield equal profit and the profit maximizing input-output bundle depends on the input and output prices. The difference between the maximum profit and what the firm earns at a different technically efficient point is the unrealized profit due to allocative inefficiency. However, given that there are alternative ways to project an inefficient input-output bundle on to the frontier, there are corresponding alternative measures of technical and (associated) allocative efficiencies.

Farrell (1957) offered a multiplicative decomposition of the overall efficiency (now described as the cost efficiency) of a firm into two distinct components - one representing its *technical efficiency* and the other its *price efficiency* (more popularly known as *allocative efficiency*). Usually, the radial input or output-orientation is chosen for projecting an inefficient point on to the frontier and the corresponding Shephard distance function is used to measure technical efficiency. However, the directional distance function introduced by Chambers, Chung, and Färe (CCF) (1996) allows the analyst to select *any* preferred direction for measuring technical efficiency.

By comparison, there are far fewer analytical models for measuring and decomposing profit efficiency. Two among the few notable papers in this area are Banker and Maindiratta (BM) (1988) and (CCF) (1998). A more recent paper by Färe, He, Li, and Zelenyuk (FHLZ) (2019) provides a unifying framework for alternative radial (Farrell-type) measures of profit efficiency. Except for BM (1988), all other papers provide an additive rather than a multiplicative decomposition of profit (in) efficiency. But, in some cases the technical efficiency component depends upon prices and/or the allocative efficiency may not lie between 0 and 1.

In this paper, we use the directional distance function as the principal analytical tool to measure technical efficiency. We also include McFadden's gauge function which leads to a multiplicative decomposition of profit efficiency where the technical efficiency factor is price independent and both technical and allocative efficiencies lie between 0 and 1. In every such case, the direction of projection of an inefficient unit on to the frontier is exogenously determined. Here we build upon a model of overall technical inefficiency from Ray (2007) further developed in Aparicio, Pastor, and Ray (APR) (2013) to show how the direction of projection is *endogenously* determined by optimization of the objective function. This direction can be used to measure the technical efficiency component of profit efficiency in a multiplicative decomposition. Like FHLZ (2019) we also consider only Farrell-type or radial variation in input and/or output vectors to measure technical efficiency.

The rest of the paper is organized as follows. Section 2 defines the production possibility set and the general assumptions about the technology. Section 3 states the profit maximization problem and identifies technical and allocative inefficiencies as two different sources of profit inefficiency. Section 4, in its different subsections, describes alternative measures of technical efficiency that correspond to different directions chosen for the directional distance function of Chambers, Chung, and Färe (CCF) (1998) along with McFadden's gauge function and an endogenous DDF based on Ray (2007). Section 5 considers the technically efficient profit that would be attained at these different efficient projections and multiplicative decomposition of profit efficiency using the different technically efficient profits to measure technical (profit) efficiency. Section 6 briefly describes the non-parametric (DEA) measures of profit efficiency and its components. Section 7 provides a simple empirical application using a small hypothetical data set from FHLZ (2019). Section 8 is the conclusion.

2. The Production Technology

Consider an industry producing *m* outputs using *n* inputs. Denote output bundles by vectors $y \in R_+^m$ and input bundles by $x \in R_+^n$. An input-output combination (x, y) is feasible if the output *y* can be produced from the input *x*. The production technology faced by the firms in the industry can be defined by the production possibility or technology set

$$T = \{(x, y): y \text{ can be produced from } x\}. \quad (1)$$

In parametric models, the set *T* is typically defined by the production function

$$y = f(x); y \in R_+,\ x \in R_+^n \quad (2)$$

in the single output case, and by the transformation function

$$F(x, y) = \alpha;\ x \in R_+^n,\ y \in R_+^m \quad (3)$$

in the multiple output case.

The corresponding technology sets are

$$T = \{(x, y): y \le f(x); x \in R_+^n; \mathrm{y} \in R_+\} \quad (4)$$

and

$$T = \{(x, y): F(x, y) \leq 0; x \in R_+^n; y \in R_+^m\} \qquad (5)$$

It is assumed that the production set is closed, bounded, and weakly monotonic in both inputs and outputs. Weak monotonicity implies

$\frac{\partial f}{\partial x_i} \geq 0, (i = 1,2, \ldots, n)$ (6a) in the single output case, and

$\frac{\partial F}{\partial x_i} \leq 0, (i = 1,2, \ldots, n)$; $\frac{\partial F}{\partial y_j} \geq 0, (j = 1, 2, \ldots, m)$ (6b) in the multiple output case.

Speaking more generally, weak monotonicity implies that if $(x^0, y^0) \in T$ and $x^1 \geq x^0$, then $(x^1, y^0) \in T$. Similarly, if $y^1 \leq y^0$, then $(x^0, y^1) \in T$. Weak monotonicity is also described as free disposability of inputs and outputs.

3. Profit Maximization and Profit Efficiency

Now suppose that the competitive market prices for outputs and inputs are *p* and *w*, respectively. A firm's profit maximization problem then becomes

$$max\, \pi = p' y - w' x$$

$$s.t. (x, y) \in T. \qquad (7)$$

For a firm producing output y^0 from input x^0 and facing the market prices (p, w) the actual profit is

$$\pi_0 = p' y^0 - w' x^0. \qquad (8)$$

Because the observed input-output bundle is feasible by default, $\pi_0 \leq \pi^*$. Further, assume that both π^* and π_0 are non-negative.[1] Thus, $0 \leq \pi_0 \leq \pi^*$.

There are two potential sources of deviation of actual profit π_0 from the maximum profit π^*. First, the actual input-output bundle (x^0, y^0) may be technically inefficient lying in the interior of the production possibility set. In that case, projecting it on to the frontier will imply either reducing inputs without lowering output (thereby lowering the cost with revenue held constant), or increasing outputs without increasing input (leading to increased revenue with cost

[1] We assume that with free exit long-run profit will not be negative.

unchanged), or some combination of both. In any of these cases, the profit will increase. Any such potential increase in profit can be ascribed to technical (in)efficiency.

Projecting an inefficient input-output bundle on to the frontier may not exhaust the potential for increasing profit, however. Even though all input-output bundles located on the frontier are technically efficient, they do not yield equal profit. There may be room for further increases in profit by moving to a different point on the frontier by changing the mix of inputs and/or outputs. Any potential increase in profit achievable through change of the mix of inputs and outputs relates to allocative (in)efficiency of the firm.

While there is no ambiguity about the actual profit $\pi_0 = p'y^0 - w'x^0$ or the optimal profit $\pi^* = p'y^* - w'x^*$, profit efficiency has been measured in the literature either by the ratio of the actual over the maximum profit or by their difference. Moreover, a decomposition of either the ratio of or the difference between the two into technical and allocative efficiencies depends on the choice of the direction of projection of any technically inefficient input-output bundle on to the frontier.

4. Alternative Measures of Technical Efficiency

4.1 Shephard Output and Input Distance Functions

Shephard (1953, 1970) defined the *output-oriented* distance function measured at an input-output bundle (*x, y*) as

$$D^y(x, y) = min\, \mu : (x, \frac{1}{\mu} y) \in T. \qquad (9)$$

This is also the radial output efficiency defined by Farrell (1957)

$$\tau_y(x, y) = \frac{1}{\varphi^*}; \varphi^* = max\, \varphi : (x, \varphi y) \in T. \qquad (10)$$

Analogously, the *input-oriented* distance function is

$$D^x(x, y) = max\, \delta : (\frac{1}{\delta} x, y) \in T. \qquad (11)$$

The input-oriented distance function is the inverse of the radial input efficiency of Farrell (1957)

$$\tau_x(x, y) = \theta^* = min\, \theta : (\theta x, y) \in T. \qquad (12)$$

Any input-output bundle (x^0, y^0) which is technically inefficient can be projected on to the efficient frontier either by proportionally scaling down the input bundle x^0 by the factor $\theta(=\frac{1}{\delta})$ or by proportionally scaling up the output bundle y^0 by the factor $\varphi(=\frac{1}{\mu})$. Note that in these two radial projections either inputs or outputs are scaled proportionally.

4.2 McFadden's Gauge Function

McFadden (1968), on the other hand, proposed the *gauge function* where one measures the equi-proportional scaling of both the input and the output vector in the same direction. For this, he first defined the negative image of the production possibility set in the netput space as

$$T^- = \{(-x, y): (x, y) \in T\} \qquad (13)$$

The gauge function is

$$H(-x, y) = \gamma^* = min\,\gamma : \left(-\frac{1}{\gamma}x, \frac{1}{\gamma}y\right) \in T^-$$

$$\Leftrightarrow min\,\gamma : (\frac{1}{\gamma}x, \frac{1}{\gamma}y) \in T. \qquad (14)$$

Hence, for any technically inefficient input bundle(x^0, y^0) located in the interior of T, $\gamma^* < 1$ and the gauge function projection will scale both x^0 and y^0 upwards by the factor $\sigma^* = \frac{1}{\gamma^*} > 1$ till it reaches the frontier.

While the radial input- or output-oriented projections continue to be the popular benchmarks for comparison for any inefficient input-output bundle that lies in the interior of the production possibility set, the direction of projection can be chosen quite arbitrarily and need not even be restricted to proportional scaling of the input or the output bundle.

An interesting point to note is that an input-output bundle that is efficient in both the input- and output-orientation may be found to be inefficient based on McFadden's gauge function. This happens when the ray through the origin intersects the production frontier twice – once at the observed input-output bundle and the second time at a point further towards the northeast. This possibility is shown below in Figure 1.

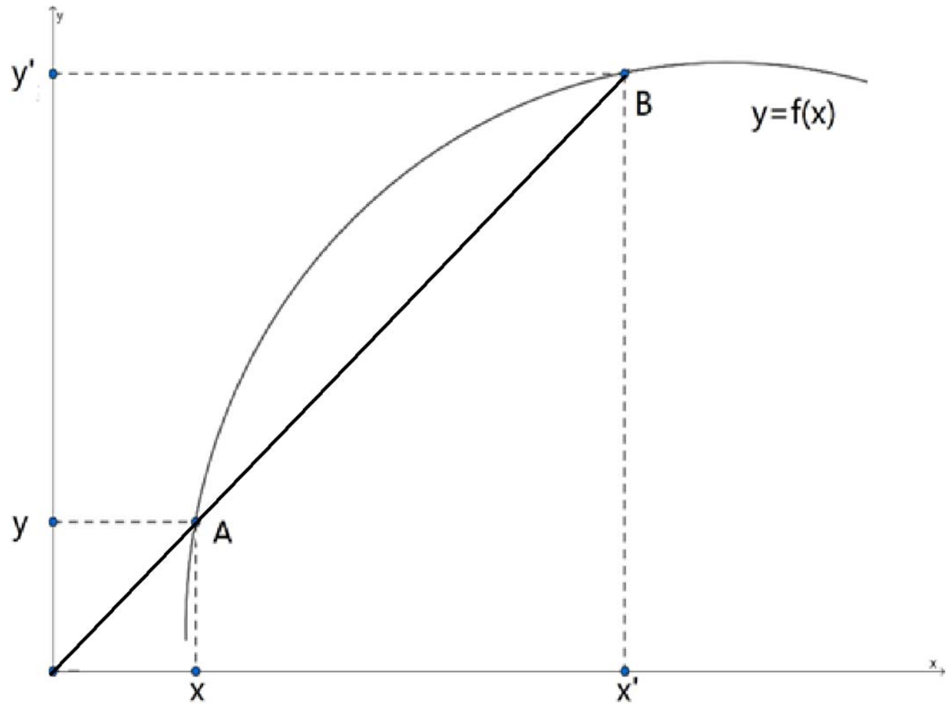

Figure 1 Gauge Function for a Technically Efficient Point

4.3 Directional Distance Function

Chambers, Chung, and Färe (CCF) (1996) introduced an efficiency measure (based on the *benefit function* of Luenberger (1992) that projects any observed input-output bundle on to the frontier in any arbitrarily chosen direction (g^x, g^y) and defined the directional distance function (DDF) as:

$$\vec{D}(x^0, y^0; g^x, g^y) = \beta^* = max\,\beta : (x^0, y^0) + \beta(-g^x, g^y) \in T$$
$$\Leftrightarrow max\,\beta : (x^0 - \beta g^x, y^0 + \beta g^y) \in T. \quad (15)$$

It is obvious that for $(g^x = 0, g^y = y^0)$, $\beta = \varphi - 1$ and $(g^x = x^0, g^y = 0)$ leads to $\beta = 1 - \frac{1}{\delta}$.

A popular direction chosen in empirical applications is $(g^x = x^0, g^y = y^0)$, The DDF then becomes

$$\vec{D}(x^0, y^0; x^0, y^0) = max\,\beta : ((1 - \beta)x^0, (1 + \beta)y^0) \in T. \quad (16)$$

Clearly, for McFadden's gauge function, $(g^x = -x^0, g^y = y^0)$ and

$$\vec{D}(x^0, y^0; x^0, y^0) = max\,\beta : ((1 + \beta)x^0, (1 + \beta)y^0) \in T. \quad (17)$$

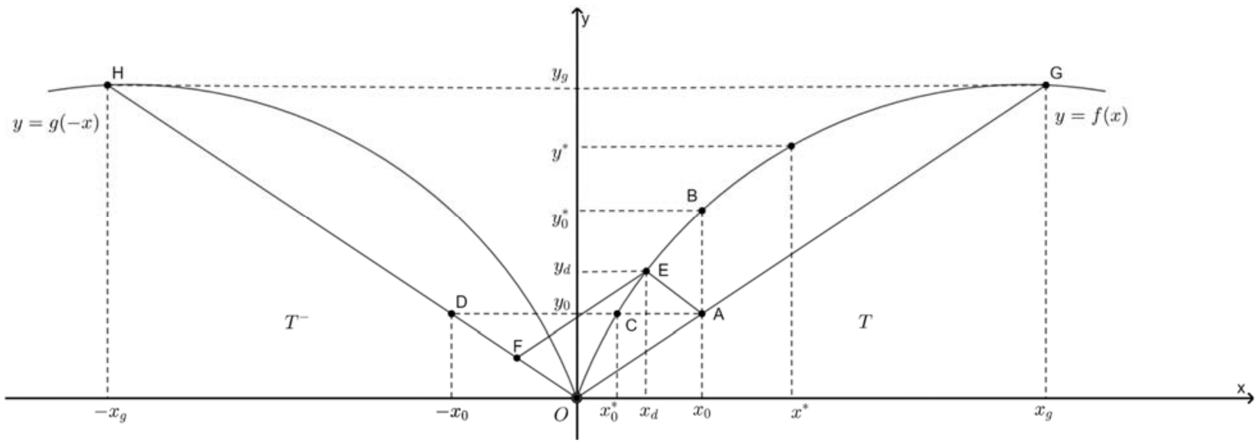

Figure 2 Alternative Technically Efficient Projections on to the Frontier

In Figure 2, point *A* represents the input-output bundle (x^0, y^0), the curve $y = f(x)$ is the production frontier and $y = g(-x)$ is its negative image in the $(-x, y)$ quadrant. The point *B* vertically above *A* is the output-oriented efficient projection $(x^0, y_0^*) = (x^0, \varphi^* y^0)$. Thus,

$$D^y(x^0, y^0) = \frac{Ax_0}{Bx_0} = \frac{Oy_0}{Oy_0^*}. \qquad (18)$$

Point *C* is the input-oriented efficient projection $(x_0^*, y^0) = (\theta^* x^0, y^0)$ and the input-oriented distance function is

$$D^x(x^0, y^0) = \frac{Ay_0}{Cy_0} = \frac{Ox_0}{Ox_0^*} = \frac{1}{\tau_x(x^0,y^0)}. \qquad (19)$$

Point *D* in the northwest quadrant represents the netput $(-x^0, y^0)$ and the point *E* on the production frontier $y = f(x)$ is the efficient projection of (x^0, y^0) in the direction $(g^x = x^0, g^y = y^0)$. The corresponding DDF is

$$\vec{D}(x^0, y^0; g^x = x^0, g^y = y^0) = \beta = \frac{AE}{OA} = \frac{OF}{OD} \qquad (20)$$

Finally, the point *G* on the curve $y = f(x)$ is the radial outward projection of the observed input-output bundle at point *A*. Corresponding to point G, point *H* is the projection of the netput bundle *D* on to the curve $y = g(-x)$ leading to the efficient input-output bundle $(x_g, y_g) = (\frac{1}{\gamma} x^0, \frac{1}{\gamma} y^0)$ and the gauge function is

$$H(x^0, y^0) = \gamma = \frac{OG}{OA} = \frac{OH}{OD}. \qquad (21)$$

Points *C, E, B,* and *G* are all located on the frontier of the production possibility set and are, therefore, technically efficient. However, as shown in Figure 3, each one of them leads to a different level of profit and in most cases, none of them yields the maximum profit. However, each one of them leads to a higher profit than $\pi_0 = py^0 - wx^0$. For points *C, E,* and *B* it is obvious because in all of these cases either cost is lower (as at *C*) or revenue is higher (as at *B*) or both (as at *E*). At the gauge function projection *G*, both input and output are bigger than at *A* and one may wonder if the cost may increase more than revenue leading to a lower profit than π_0. But, because $(x_g, y_g) = \sigma(x^0, y^0)$ and $\sigma > 1$ profit at point G is $\sigma\pi_0 > \pi_0$ so long as $\pi_0 > 0$.

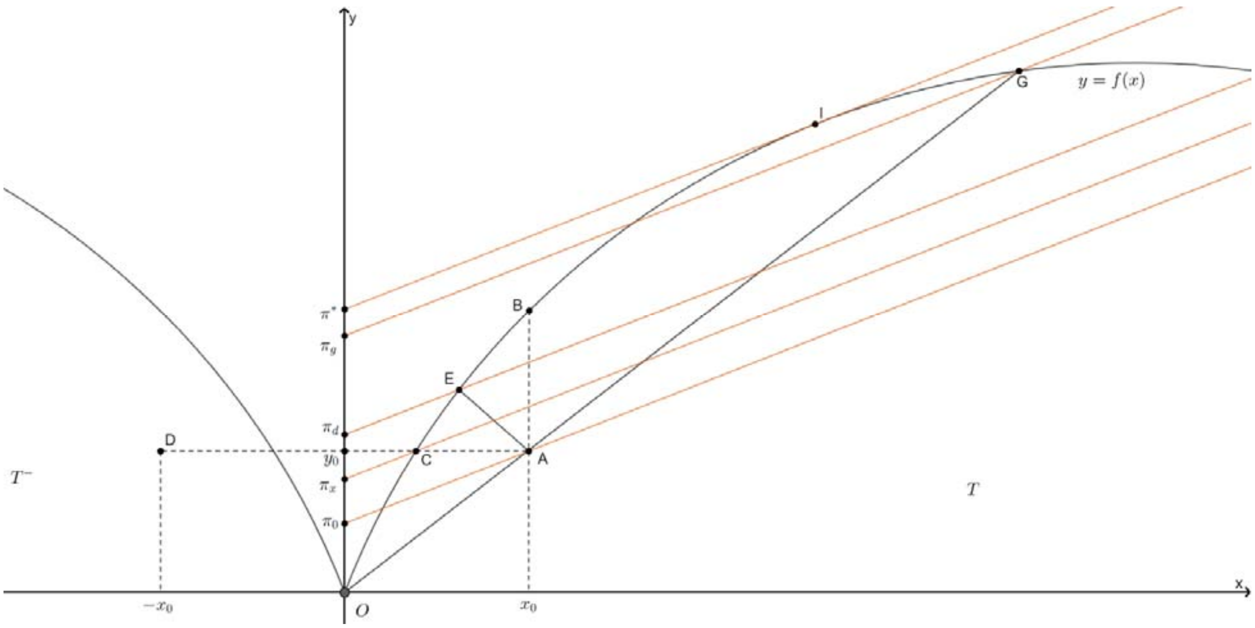

Figure 1 Profit Functions Based on Alternative Technically Efficient Projections

4.4 <u>Endogenous Directional Distance Function</u>

A somewhat different version of the directional distance function in (16) is implied by the *overall (in)efficiency* measure in Ray (2007):

$$max\, \eta = \varphi - \theta: (\theta x^0, \varphi y^0) \in T. \qquad (22)$$

Subsequently, Aparicio, Pastor, and Ray (APR) (2012) set up (21) as

$$max\, \beta_x + \beta_y: ((1 - \beta_x)x^0, (1 + \beta_y)y^0) \in T. \qquad (23)$$

It is clear that under the added constraints $\beta_x = \beta_y, (\varphi = 1 + \beta; \theta = 1 - \beta)$ the problems in (22) and (23) reduce to the DDF in (16). However, without these restrictions there is considerable flexibility in choosing the direction of projection on to the frontier.

Consider, for simplicity, the 1-input 1-output case with the production function $y = f(x)$. The problem in (22), then becomes

$$max\, \eta = \varphi - \theta: \varphi y^0 \leq f(\theta x^0). \qquad (24)$$

For a given input-output pair (x^0, y^0) the constraint $\varphi y^0 \leq f(\theta x^0) \Leftrightarrow \varphi \leq \frac{f(\theta x^0)}{y^0} \equiv g(\theta)$

Because the efficient projection will be on the frontier, the constraint will be binding and can be treated as an equation. The Lagrangian for the constrained optimization problem is

$$L = \varphi - \theta + \lambda[g(\theta) - \varphi] \qquad (25)$$

The first order conditions for a maximum are

$$\frac{\partial L}{\partial \varphi} = 1 - \lambda = 0;$$

$$\frac{\partial L}{\partial \theta} = -1 + \lambda g' = 0;$$

$$\frac{\partial L}{\partial \lambda} = g(\theta) - \varphi = 0; \qquad (26)$$

Thus, at the optimal pair of (θ, φ) $\frac{d\varphi}{d\theta} = g'(\theta) = 1$ and the tangent to the curve $\varphi = g(\theta)$ is parallel to the 45°-line.

One potential problem is that at the optimal solution of (22), φ^* may be less than 1, in which case, for some price vectors (p, w), one may find

$$\varphi^* p y^0 - \theta^* w x^0 = \varphi^*(p y^0 - w x^0) + (\varphi^* - \theta^*) w x^0 < p y^0 - w x^0.$$

This would imply that the optimal profit at the technically efficient point is less than the actual profit at an inefficient point. An intuitive explanation of this strange result is quite simple. When $1 > \varphi^* > \theta^*$, both output and input are scaled down. But if the relative price of input $(\frac{w}{p})$ is quite low, reduction in cost cannot make up for the loss of revenue as the firm is projected on to the technically efficient point. As a result, improving its technical efficiency by moving it to the frontier lowers profit. Such a paradoxical result can be eliminated by imposing the additional restriction $\varphi \geq 1$ in (22).[2] If the lower bound on φ proves to be a nonbinding constraint, the solution based on (26) remains optimal. Otherwise, one needs to set $\varphi = 1$ and the optimal value

[2] This amounts to explicitly imposing non-negativity restriction on β_y. But if θ exceeds 1, β_x in (22) can still be negative.

of θ is $g^{-1}(\varphi)$ at $\varphi = 1$. In such cases, $\eta = 1 - \theta$ and maximizing η is the same as minimizing θ so that the optimal endogenous direction is simply the input-oriented projection.

Derivation of the endogenous optimal direction for projection in (22) is explained below in Figures 4a-4b. Point *A* in Figure 4a represents the observed input-output bundle. Points *B* $(\theta^* x_0, y_0)$ and C $(x_0, \varphi^* y_0)$ are its standard input- and output-oriented projections on to the frontier shown by the curve $y = f(x)$.

Unlike in Figure 4a, where the two axes measure the input and output *quantities* *x* and *y*, in Figure 4b, the axes measure the *scale* of the input and output treating x^0 and y^0 as their respective units of measurement. Thus, at the actual input-output bundle (x^0, y^0), $\theta = \varphi = 1$. In this panel, the production frontier is redrawn as

$$\varphi y^0 = f(\theta x^0) \Leftrightarrow \varphi = g(\theta | x^0, y^0). \qquad (27)$$

Now consider the equal (in)efficiency lines $\eta = \varphi - \theta \Leftrightarrow \varphi = \eta + \theta$ parallel to the 45°-line. Each point on a particular line has the same overall (in)efficiency as point *Q* (i.e., the observed input-output pair). The intercept of the line measures the inefficiency $(\eta = \varphi - \theta)$ of *Q*. Point *R* is where one of these lines is tangent to the line $\varphi = g(\theta)$. Thus the optimal solution to the problem in (27) is $(\theta = \theta_R, \varphi = \varphi_R)$ in Figure 4b. The corresponding point in Figure 4a in the *x*-*y* plane is $(x_R, y_R) = (\theta_R x^0, \varphi_R y^0)$ shown by the point *D* on the curve $y = f(x)$. The profit at this endogenous directional projection is the intercept of the iso-profit line

$$\pi_R = p y^R - w x^R = p \varphi_R y^0 - w \theta_R x^0 \qquad (28)$$

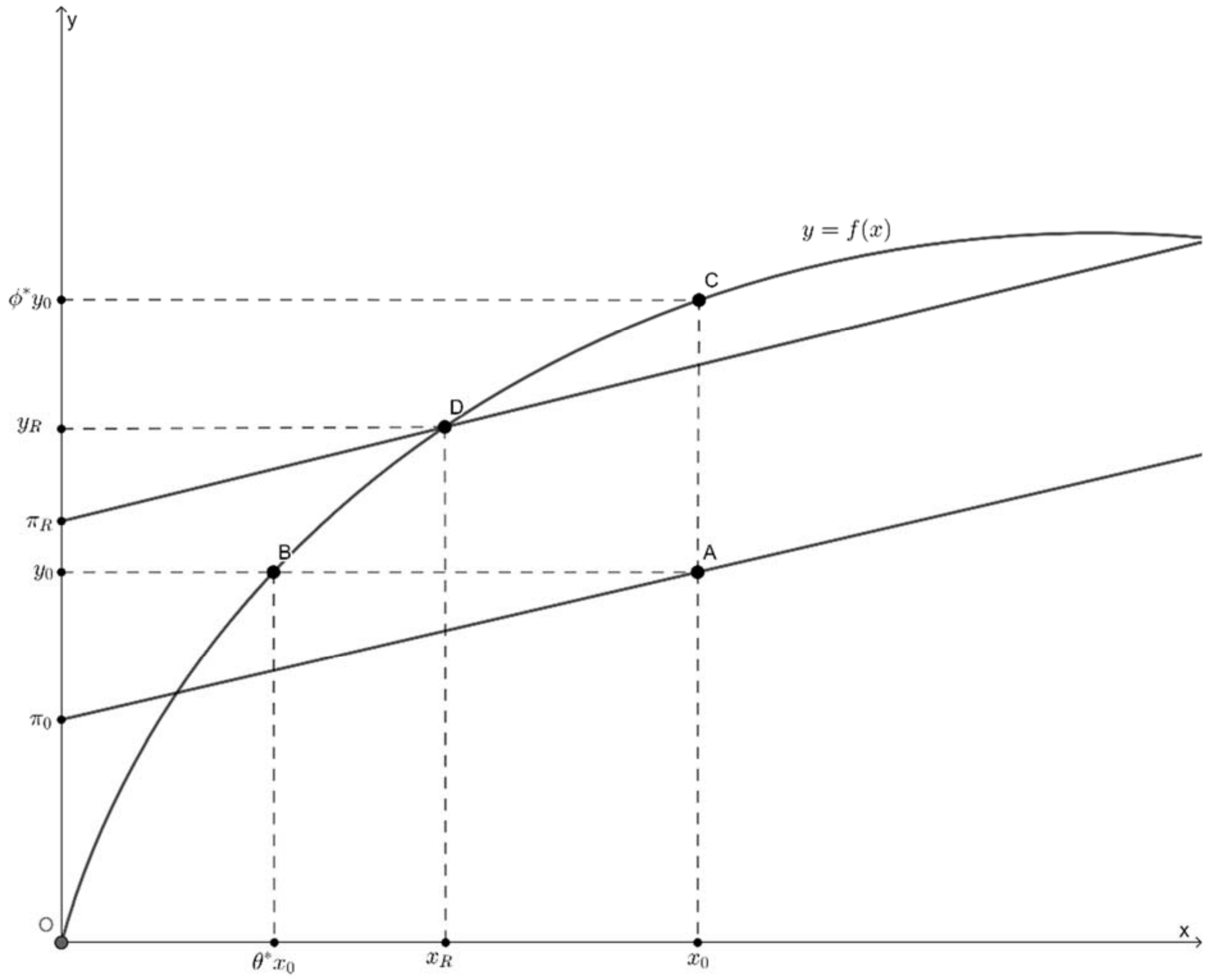

Figure 2a Endogenous Direction of Projection

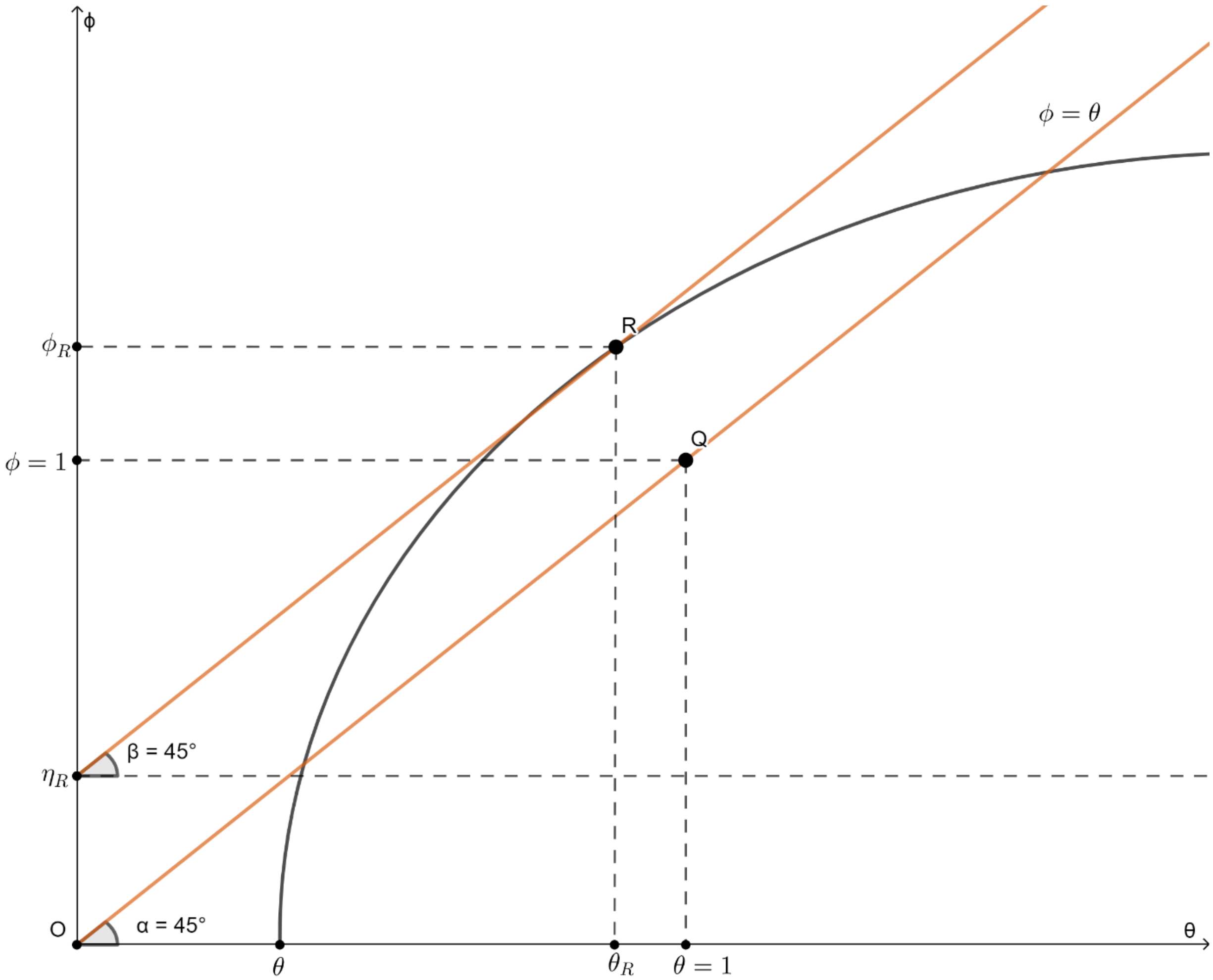

Figure 4b Finding the optimal direction of projection

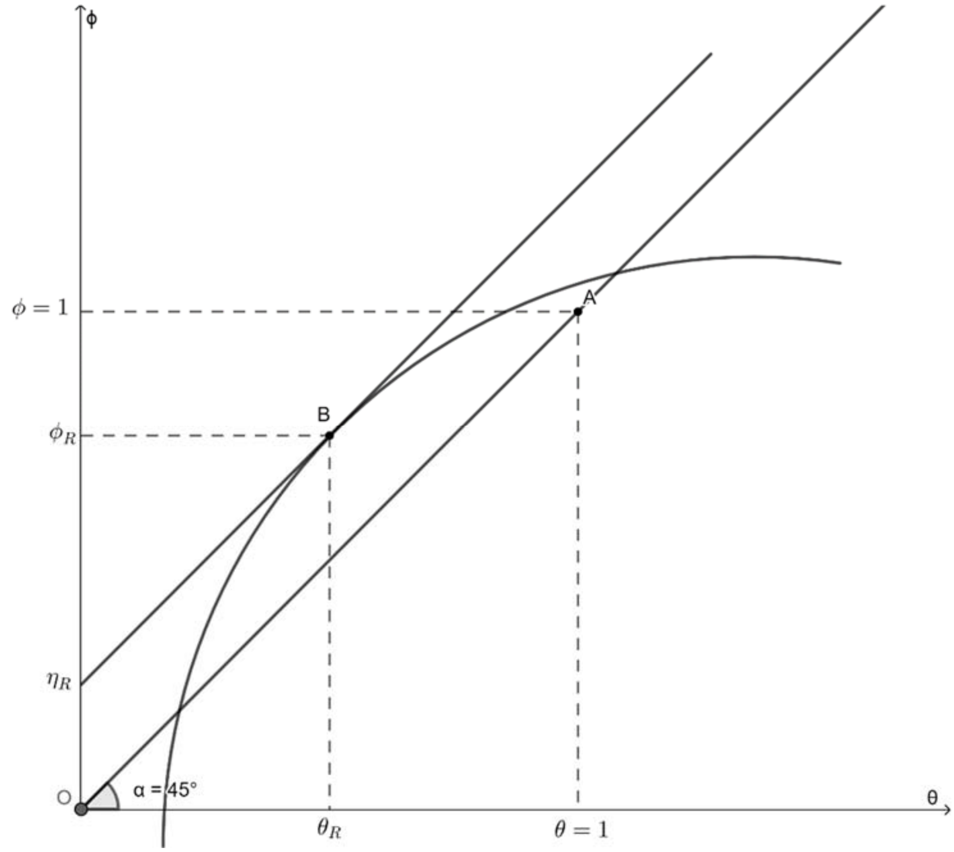

Figure 3a Optimal Endogenous Projection

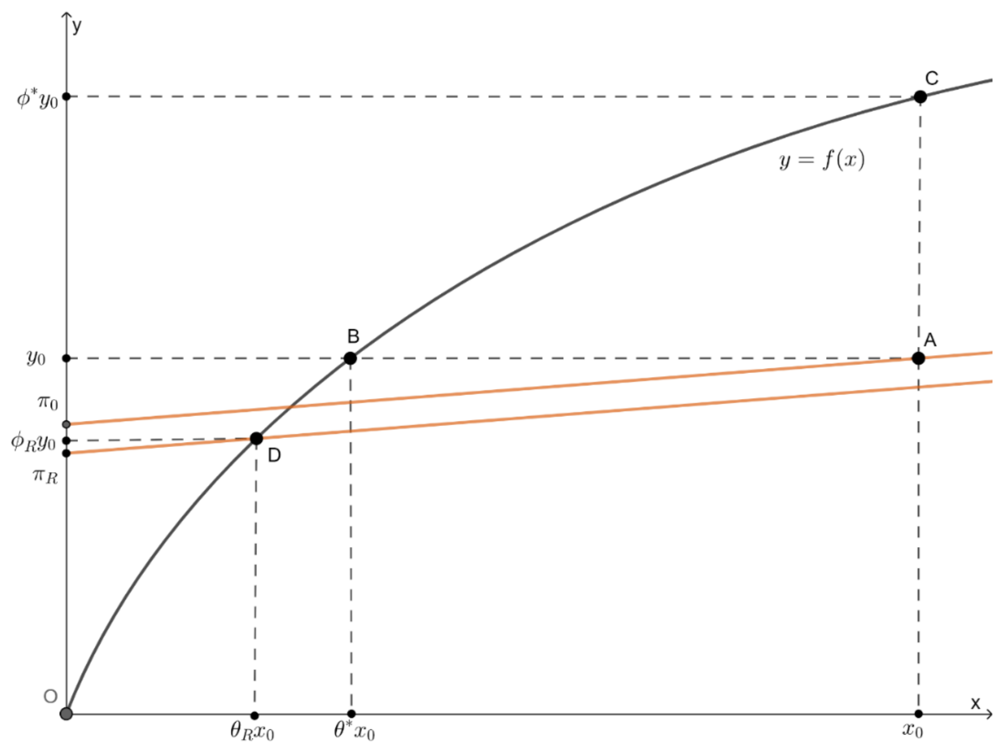

Figure 5b Profit at Optimal Endogenous Projection (D)

Figures 5a-5b illustrate the case where the profit at the technically efficient input bundle is lower than the actual profit at the inefficient input-output bundle.

In Figure 5a, point *A* ($\theta = \varphi = 1$) corresponds to the actual input-output bundle (x^0, y^0) and *B* is its optimal projection based on the LP problem in (23). Note that *B* is towards the southwest of *A* signifying reduction in both input and output (scale). The corresponding technically efficient input-output bundle ($x_R = \theta_R x^0, y_R = \varphi_R y^0$) is shown by point *D* in Figure 5b.

The slope of the parallel iso-profit lines through *A* and *D* corresponds to the relative price of the input ($\frac{w}{p}$). Because the profit line through *A* is higher than the one through *D*, the lower bound $\varphi \geq 1$ will become binding and the optimal direction of projection will be input-oriented leading to point *B*.

4.5 <u>An Example of the Optimal Endogenous Direction of Projection</u>

Consider a simple example. Suppose that

$$f(x) = 2\sqrt{x}\,;\ (x_0 = 9, y_0 = 4). \qquad (29)$$

Then, the production function in Figure 4b would be

$$4\varphi = 6\sqrt{\theta} \Rightarrow \varphi = \frac{3}{2}\sqrt{\theta}. \qquad (30)$$

The problem in (22) then is

$$\max \eta = \varphi - \theta$$
$$s.t. \varphi = \frac{3}{2}\sqrt{\theta} = g(\theta) \qquad (31)$$

Using the first order conditions for a maximum we get

$$\frac{d\varphi}{d\theta} = \frac{3}{4\sqrt{\theta}} = 1 \Rightarrow \theta_R = \frac{9}{16} = 0.5625$$
$$\varphi_R = g(\theta_R) = \frac{3}{2}\sqrt{\theta} = \frac{9}{8} = 1.125$$
$$\eta_R = \varphi_R - \theta_R = 1.125 - 0.5625 = 0.5625 \qquad (32)$$

The corresponding technically efficient input-output bundle is

$$(x_R = \theta_R x^0 = 5.0625; y_R = \varphi_R y^0 = 4.5). \qquad (33)$$

It should be noted in this context that the optimal solution of the problem in (23) will not necessarily lead to $\theta_R < 1$. Suppose that in the example (29), the actual input-output bundle was $(x^0 = 4, y^0 = 1)$. In that case, $\varphi = g(\theta)$ would be $\varphi = 4\sqrt{\theta}$ and setting $\frac{d\varphi}{d\theta} = 1$ would lead to $\theta_R = 4$ and $\varphi_R = 8$. In this case, the technically efficient projection would be $(x_R = 16, y_R = 8)$. Hence, the direction of projection is not always towards the northwest of the observed input-output bundle.

Two other papers which also consider endogenous directions for a DDF are Färe, Grosskopf, and Whittaker (FGW) (2013) and Zofio, Pastor, and Aparicio (ZPA) (2013). FGW (2013) consider an output-oriented DDF (under CRS) for a two-output case and end up with a model similar to APR (2013) except that in their model $g^x = 0$ (the null vector) and their output direction is g^y, the 45°-line modified endogenously by the ratio of the output-specific expansion factors. ZPA (2013), on the other hand, start with the profit-maximizing input-output bundle (x^*, y^*) as the benchmark for projection on to the frontier. If the observed input-output bundle is already on the frontier but is not the profit-maximizing bundle, they interpret any profit inefficiency as allocative. That is consistent with either the multiplicative or the additive decomposition where there is no technical inefficiency. However, if (x^0, y^0) is an interior point, they argue that any profit inefficiency is technical inefficiency. A problem with this interpretation is that such technical efficiency cannot be measured solely from input-output data without information on input and output prices. By contrast, all of the other measures of *technical* efficiency – input oriented, output oriented, DDF, McFadden's gauge function, or the overall efficiency – can be measured from input-output data alone.

5. Technically Efficient Profit

The firm's profit at the different technically efficient projections of the inefficient input-output bundle can be measured as:

$\pi_x = py^0 - \theta^* wx^0$ at the input-oriented projection; (34)

$\pi_y = \varphi^* py^0 - wx^0$ at the output-oriented projection; (35)

$$\pi_d = (1+\beta^*)py^0 - (1-\beta^*)wx^0$$
$$= py^0 - wx^0 + \beta^*(py^0 + wx^0) \quad (36)$$

at the DDF projection, and

$\pi_g = \frac{1}{\gamma^*}(py^0 - wx^0) = \alpha\pi_0$ at the gauge function projection. (37)

5.1 Additive and Multiplicative Decomposition of Profit Efficiency

Any shortfall of the actual profit from the maximum implies inefficiency in the form of unrealized profit. We have seen that inefficiency arises both when the selected input-output bundle lies in the interior (rather than the frontier) of the production possibility set and when a suboptimal point on the frontier is chosen. Thus, an appropriate additive decomposition of the unrealized profit of a firm is

$$\Delta = \pi^* - \pi^0 = (\pi^* - \pi_T) + (\pi_T - \pi^0) \quad (38)$$

The difference between the technically efficient profit (π_T) measured by any one of the expressions in (28) or (34) through (37) and the actual profit π_0 is a measure of the lost profit that can be ascribed to technical inefficiency. But as noted above, while the alternative technically efficient bundles depend only upon the production function $y = f(x)$ or the transformation function $F(x, y) = 0$ and the observed input-output bundle (x^0, y^0) the profit-maximizing input-output bundle (x^*, y^*) will depend also upon the applicable input-output prices. The difference between the maximum profit $\pi^*$and any chosen technically efficient profit π_T is due to allocative inefficiency.

A major limitation of the difference Δ as a measure of profit efficiency is that it ignores the scale of operation of the firm. Thus, a firm with $\pi_0 = 2, \pi^* = 5$ resulting in $\Delta = 3$ is regarded as more efficient than another firm with $\pi_0 = 100, \pi^* = 104$ and $\Delta = 4.$

Farrell (1957) offered a multiplicative decomposition of cost or economic efficiency into two distinct factors as

$$\frac{C^*}{C_0} = \left(\frac{C_T}{C_0}\right)\left(\frac{C^*}{C_T}\right) \quad (39)$$

where

$$C^* = min\, w'x: (x, y^0) \in T \quad (40)$$

and

$$C_T = \theta^* w'x^0 = \theta^* C_0 \quad (41)$$

where

$$\theta^* = min\, \theta : (\theta x^0, y^0) \in T. \quad (42)$$

Thus, the two constituent factors of cost efficiency (*CE*) are

Technical Efficiency $TE = \frac{C_T}{C_0} = \theta^*$ (43) and

Allocative Efficiency $AE = \frac{C^*}{C_T}$ (44)

It may be noted that cost efficiency *CE* and its components technical efficiency (*TE*) and allocative efficiency (*AE*) all lie between 0 and 1.

In a comparable way, one can define and decompose

Profit Efficiency: $PE = \frac{\pi_0}{\pi^*} = \left(\frac{\pi_0}{\pi_T}\right)\left(\frac{\pi_T}{\pi^*}\right)$ (45)

In (45) above the first factor on the right corresponds to technical efficiency and the other is allocative efficiency.

BM (1988) used $\pi_x$from (37) for π_T in (45) to get

$$PE_x = \left(\frac{py^0 - wx^0}{py^0 - \theta^* wx^0}\right)\left(\frac{py^0 - \theta^* wx^0}{py^* - wx^*}\right) = TE_x . AE_x \qquad (46)$$

While both components of PE_x in (46) above will lie between 0 and 1, the technical efficiency component TE_x is price dependent and unlike Farrell's input-oriented technical efficiency does not depend on the production frontier alone.

FHLZ (2019) define a Farrell output-oriented profit efficiency as

$$PE_0 = max\ \varphi$$

$$s.t.\ \varphi py^0 - wx^0 \leq py - wx$$
$$(x, y) \in T \qquad (47)$$

But $\varphi py^0 - wx^0 \leq py - wx; (x, y) \in T \Rightarrow \varphi py^0 - wx^0 \leq \pi^*$. Hence, at the optimum

$$\varphi py^0 = \pi^* + wx^0$$

$$\Rightarrow \varphi = \frac{\pi^* - (py^0 - wx^0) + py^0}{py^0}$$

$$\Rightarrow \varphi - 1 = \frac{\pi^* - \pi_0}{R_0}; R_0 = py^0 \qquad (48)$$

On the other hand, if one uses the ratio of actual profit over maximum profit and uses π_y from (38) to measure π_T, one gets

$$PE_y = \left(\frac{\pi_0}{\pi_y}\right)\left(\frac{\pi_y}{\pi^*}\right) = \left(\frac{py^0 - wx^0}{\varphi^* py^0 - wx^0}\right)\left(\frac{\varphi^* py^0 - wx^0}{py^* - wx^*}\right) = TE_y . AE_y \qquad (49)$$

If one defines the actual 'return on outlay' as $\rho_0 = \frac{py^0}{wx^0}$, in the factorization in (49), technical efficiency becomes

$$TE_y = \frac{\rho_0 - 1}{\varphi^* \rho^0 - 1} \qquad (50)$$

Note that so long as $\pi_0 > 0, \rho_0 > 1$ and $0 < TE_y < 1$. It should be noted, however, that for any $(x^0, y^0), \rho_0 = \frac{py^0}{wx^0}$ depends on the prices.

In an analogous way, TE_x in (46) can be expressed as

$$TE_x = \frac{py^0 - wx^0}{py^0 - \theta^* wx^0} = \frac{\rho_0 - 1}{\rho_0 - \theta^*}. \qquad (51)$$

CCF (1998) proposed a measure of what they describe as Nerlovian efficiency based on a directional distance function as

$$NE = \frac{\pi^* - \pi_0}{wg^x + pg^y} = \frac{\Delta}{wg^x + pg^y} \qquad (52)$$

They provide an additive decomposition of Nerlovian efficiency as

$$NE = \frac{\pi^* - \pi_0}{wg^x + pg^y} = \frac{(\pi_T - \pi_0) + (\pi^* - \pi_T)}{wg^x + pg^y}$$

$$= \frac{(\pi_T - \pi_0)}{wg^x + pg^y} + \frac{(\pi^* - \pi_T)}{wg^x + pg^y}$$

$$= TE_N + AE_N \tag{53}$$

Using the direction ($g^x = x^0, g^y = y^0$), and $\pi_T = \pi_d$ from (39), one gets

$$NE = \frac{(py^* - wx^*) - (py^0 - wx^0)}{py^0 + wx^0};$$

$$TE_N = \beta^* = \vec{D}(x^0, y^0; g^x = x^0, g^y = y^0);$$

$$AE_N = NE - TE_N \tag{54}$$

In this additive decomposition, TE_N is independent of prices but AE_N may exceed 1. Also, the interpretation of the sum of the actual revenue and cost as a measure of the size of the firm is not intuitively obvious.

Alternatively, in a geometric decomposition of profit efficiency

$$PE_d = \left(\frac{\pi_0}{\pi_d}\right)\left(\frac{\pi_d}{\pi^*}\right)$$

$$= \left(\frac{py^0 - wx^0}{py^0 - wx^0 + \beta^*(py^0 + wx^0)}\right)\left(\frac{py^0 - wx^0 + \beta^*(py^0 + wx^0)}{py^* + wx^*}\right)$$

$$= TE_d . AE_d \tag{55}$$

Here

$$TE_d = \frac{py^0 - wx^0}{py^0 - wx^0 + \beta^*(py^0 + wx^0)}$$

$$= \frac{\rho_0 - 1}{(1 + \beta^*)\rho_0 - (1 - \beta^*)}$$

$$= \frac{\rho_0 - 1}{(\rho_0 - 1) + \beta^*(\rho_0 + 1)} \tag{56}$$

Again, $\beta^* \geq 0, \rho_0 > 1$ ensure that $0 < TE_d < 1$. However, the technical efficiency component in this decomposition is not independent of prices.

If we use McFadden's gauge function for technical efficiency, the corresponding multiplicative decomposition of profit efficiency will be

$$PE = \frac{\pi_0}{\pi^*} = \left(\frac{\pi_0}{\pi_g}\right)\left(\frac{\pi_g}{\pi^*}\right)$$

$$= \left(\frac{py^0 - wx^0}{\frac{1}{\gamma *}(py^0 - wx^0)}\right)\left(\frac{\frac{1}{\gamma *}(py^0 - wx^0)}{py^* - wx^*}\right)$$
$$= \gamma^* \left(\frac{py^0 - wx^0}{\gamma^*(py^* - wx^*)}\right)$$
$$= TE_g . AE_g \qquad (57)$$

It may be noted that in (57) the technical efficiency factor is independent of prices and also that both factors lie between 0 and 1.

As noted before, in all of the decompositions of profit efficiency considered above, the direction of technically efficient projection of any inefficient input-output bundle is preassigned by the analyst. If, instead, one uses the optimization model of Ray (2007) and uses π_R from (27) for π_T one gets

$$PE = \left(\frac{\pi_0}{\pi_R}\right)\left(\frac{\pi_R}{\pi^*}\right)$$
$$= \left(\frac{py^0 - wx^0}{\varphi_R py^0 - \theta_R wx^0}\right)\left(\frac{\varphi_R py^0 - \theta_R wx^0}{py^* - wx^*}\right)$$
$$= TE_R . AE_R \qquad (58)$$

The technical efficiency factor can be rewritten as

$$TE_R = \frac{py^0 - wx^0}{\varphi_R py^0 - \theta_R wx^0}$$
$$= \frac{\rho_0 - 1}{\varphi_R \rho_0 - \theta_R} \qquad (59)$$

As argued before, if $\varphi_R > 1, \pi_R > \pi_0$, $0 < TE_R < 1$ so long as $\pi_0 > 0$ so that $\rho_0 > 1$. Otherwise, $\pi_R = \pi_x$ and $TE_R = TE_x \Rightarrow AE_R = AE_x$.

6. Nonparametric Approximation of the Technology

Consider the input-output data set $\mathcal{D}=\{(x^j, y^j); x^j \in R_+^n, y^j \in R_+^m, j = 1,2,...,N\}$ for N firms in an industry. Based on the assumptions of convexity and closedness of the production possibility set and free disposability of inputs and outputs, a nonparametric approximation of the technology is the free disposal convex hull of the data set:

$$S = \{(x, y): x \geq \sum_{j=1}^{N} x^j; y \leq \sum_{j=1}^{N} y^j; \sum_{j=1}^{N} \lambda_j = 1; \lambda_j \geq 0; j = 1,2,...,N\} \quad (60)$$

For any observed input-output bundle $(x^k, y^k) \in \mathcal{D}$ facing output and input prices (p^k, w^k), the actual $\varphi^* = max\, \phi$ profit is $\pi_o^k = p^k y^k - w^k x^k$ and maximum profit is

$$\pi^*(p^k, w^k) = max\, p^k y - w^k x$$
$$s.t. (x, y) \in S \qquad (61)$$

For the variety of technical efficiencies and the corresponding technically efficient profit (for any firm k in the sample) one needs to solve the relevant optimization problem from below:

Output-oriented Efficiency

$$\varphi_k^* = max\,\varphi$$
$$s.t.(x^k, \varphi y^k) \in S \qquad (62)$$

Input-oriented Efficiency

$$\theta_k^* = min\,\theta$$
$$s.t.(\theta x^k, y^k) \in S \qquad (63)$$

Directional Distance Function

$$\beta_k^* = max\,\beta$$
$$s.t.\big((1-\beta)x^k, (1+\beta)y^k\big) \in S \qquad (64)$$

McFadden's Gauge Function

$$\gamma_k^* = min\,\gamma$$
$$s.t.\left(\frac{1}{\gamma}x^k, \frac{1}{\gamma}y^k\right) \in S \qquad (65)$$

Endogenous Directional Distance Function

$$\eta_k^* = \varphi_k^* - \theta_k^* = max\,\varphi - \theta$$
$$s.t.\varphi \geq 1; (\theta x^k, \varphi y^k) \in S \qquad (66)$$

Once the LP problems in (61) - (66) are solved using the input-output data from $\mathcal{D}$, profit efficiency and the technical and allocative efficiency components can be measured for individual firms.

6. An Empirical Example

We use a hypothetical 2-input 2-output data set for nine firms generated by FHLZ (2019; page 192) reproduced in Table 1 below. It should be pointed out that, at their input and output vectors of price and quantity, three firms (#5, #8, and #9) would earn negative profit ($\pi_0 < 0$) while another firm (#7) would earn zero profit. Thus, profit efficiency measured by the ratio of actual and maximum profit would be meaningless for these firms. We can, however, obtain the various measures of technical efficiency based on the alternative directional distance functions (as described above) for all nine observations based upon their input and output quantity data. We can also get meaningful measures of Nerlovian efficiency and its decomposition as in CCF (1998) even when the actual profit is zero or negative.

Table 1: Raw Data

Measurement	Notation	Firm								
		1	2	3	4	5	6	7	8	9
Output 1	y1	486	200	400	300	200	440	421.88	450	0
Output 2	y2	486	600	400	450	400	480	421.88	500	235
Input 1	x1	9	8	9	9	10	7	7.5	4	3
Input 2	x2	9	10	10	9	15	10	7.5	20	5
Output Price 1	p1	1	1	1	1	1	1	1	1	1

Output Price 2	p2	1	1	1	1	1	1	1	1	1
Input Price 1	w1	27	27	27	27	27	27	56.25	56.25	56.25
Input Price 2	w2	27	27	27	27	27	27	56.25	56.25	56.25
Actual Revenue	R_0	972	800	800	750	600	920	843.76	950	235
Actual Cost	C_0	486	486	513	486	675	459	843.75	1350	450

The different technical efficiency measures based only upon the input-output data and derived using alternative directional distance functions are reported in Table 2. Firms #1, #2, #6, #7, and #8 are found to be technically efficient in every orientation. Their input-, output-, and gauge efficiencies are all equal to 1. Accordingly, the DDF and the overall (in)efficiencies are 0. Firm #9 presents an interesting case, where the input- and output-oriented efficiencies are 1 and correspondingly the DDF equals 0 implying that there is no potential for increasing output and reducing input simultaneously. However, the value of the McFadden gauge function is 0.4 implying that both the output and the input vectors can be scaled up by a factor of 2.5. According to the endogenous DDF, the output vector should be optimally scaled up by a factor of 2.468 while the input vector scaled up by a factor of 2.4. This is quite close to the value implied by the gauge function. Note that the endogenous DDF for firm #9 shows $\eta^* = (\varphi^* - 1) + (1 - \theta^*) =$ 0.068 implying an overall efficiency of 0.932. The remaining three firms – #3, #4, and #5 – are technically inefficient by all measures. Firm #5 has significantly lower efficiency than firms #3 and #4. Firm #4 performs better than #3 by every measure of technical efficiency.

Table 2: Technical Efficiency

Measurement	Notation	Firm								
		1	2	3	4	5	6	7	8	9
Input-oriented	τ_x	1	1	0.789	0.877	0.54	1	1	1	1
Output-oriented	τ_y	1	1	0.823	0.871	0.706	1	1	1	1
Directional Distance	β*	0	0	0.124	0.072	0.31	0	0	0	0
McFadden's Gauge	1/γ*	1	1	0.823	0.838	0.706	1	1	1	0.4
Endogenous Directional Distance	η*	0	0	0.249	0.153	0.67	0	0	0	0.068
	Φ*	1	1	1.067	1.101	1.405	1	1	1	2.468
	Θ*	1	1	0.818	0.948	0.736	1	1	1	2.4
	1-η*	1	1	0.751	0.847	0.33	1	1	1	0.932

Table 3 shows the actual and maximum profit along with the profits at the different directional projections of the input-output bundles on to the frontier. As noted before, the actual profit(π_0) is negative for firms #5, #8, and #9 and zero for firm #7. By comparison, the maximum profit (π^*) is zero for firms #7, #8, and #9 and equals 486 for all other firms. Given that firms #1, #2, #6, #7, and #8 are found to be technically efficient by every criterion, their technically efficient profits (π_T) are equal to their actual profits (even when for some of them these are negative). For firm #9, π_g is even more negative than π_0.

For firms #3 and #4, profit at the output-oriented projection (π_y) is higher than at other technically efficient projections while for firm #5 profit at the endogenous projection (π_R) is the highest among all technically efficient projections. Also, for firm #5, except for McFadden's gauge projection[3], profit is positive at all technically efficient projections even though the actual profit (π_0) is negative.

Table 3: Profit Table

Measurement	Notation	Firm								
		1	2	3	4	5	6	7	8	9
Actual Profit	π_0	486	314	287	264	-75	461	0	-400	-215
Maximum Profit	π*	486	486	486	486	486	486	0	0	0
Input-oriented Profit	π_x	486	314	395.01	323.69	235.19	461	0	-400	-215
Output-oriented Profit	π_y	486	314	459	375.13	175.15	461	0	-400	-215
DDF Profit	π_d	486	314	450.19	353.2	320.43	461	0	-400	-215
McFadden's Gauge Profit	π_g	486	314	348.71	315.05	-106.3	461	0	-400	-537.5
Endogenous Directional Distance Profit	π_r	486	314	433.92	364.96	235.19	461	0	-400	-215
Profit Efficiency	PE	1	0.646	0.591	0.543	N/A	0.949	N/A	N/A	N/A

Tables 4 and 5 show the (profit-based) technical and allocative efficiency components from a multiplicative decomposition of profit efficiency under alternative directional projections. The technical efficiency measures from Table 4 are uniformly lower than the price-independent measures shown in Table 2 except in the case of McFadden's gauge function.

As noted before, profit efficiency of a firm measured by the ratio of its actual profit (π_0) to the maximum profit($\pi^* = \pi(p, w)$) is meaningful only when the actual profit is positive. In fact, 4 out of the 9 firms in this example show zero or negative profit and we cannot get sensible profit-oriented technical and allocative efficiency measures for them. But measurement and an additive decomposition of the Nerlovian efficiency is always possible because the difference($\pi^* - \pi_0$) is always non-negative (even when the actual profit is negative) and the denominator ($py^0 + wx^0$) is strictly positive.

Table 4: Profit - Oriented Technical Efficiency

Measurement	Notation	Firm								
		1	2	3	4	5	6	7	8	9
Input-oriented	TE_x	1	1	0.727	0.816	N/A	1	1	1	1

[3] The gauge function simply expands the input and output vectors radially. Hence, if the actual profit is negative, so is the profit at the technically efficient projection.

Output-oriented	TE_y	1	1	0.625	0.704	N/A	1	1	1	1
Directional Distance	TE_n	1	1	0.638	0.747	N/A	1	1	1	1
McFadden's Gauge	TE_g	1	1	0.823	0.838	0.706	1	1	1	0.4
Endogenous Directional Distance	TE_r	1	1	0.661	0.723	N/A	1	1	1	1

Table 5: Profit -Oriented Allocative Efficiency

Measurement	Notation	Firm								
		1	2	3	4	5	6	7	8	9
Input-oriented	AE_x	1	0.646	0.813	0.666	N/A	0.949	1	N/A	N/A
Output-oriented	AE_y	1	0.646	0.944	0.772	N/A	0.949	1	N/A	N/A
Directional Distance	AE_n	1	0.646	0.926	0.727	N/A	0.949	1	N/A	N/A
McFadden's Gauge	AE_g	1	0.646	0.718	0.648	N/A	0.949	1	N/A	N/A
Endogenous Directional Distance	AE_r	1	0.646	0.893	0.751	N/A	0.949	1	N/A	N/A

Table 6 shows the Nerlovian efficiency (NE) and its additive technical (TE) and allocative (AE) components for all nine firms in the example. One needs to remember that a higher value of NE implies a bigger shortfall from the maximum achievable profit and hence a lower efficiency of the unit. As explained in CCF (1998), the technical efficiency is the same as the DDF and corresponds to what we have reported as β^* in Table 2. In this table, it is to be interpreted as unachieved profit as a proportion of the sum of the revenue and the cost of a firm. For example, firm #5 has an actual profit of -75. This implies a shortfall of 561 from the maximum profit of 486. This is 44% of the sum of its revenue (600) and cost (675). The DDF implies that it could increase outputs and at the same time reduce inputs by 31% thereby increasing profit by 395.35. The remaining 165.75 is ascribed to allocative inefficiency.

Table 6 Nerlovian Efficiency

Measurement	Notation	Firm								
		1	2	3	4	5	6	7	8	9
Nerlovian Efficiency	NE	0	0.134	0.152	0.18	0.44	0.018	0	0.174	0.314
DDF TE	TE_n	0	0	0.124	0.072	0.31	0	0	0	0
DDF AE	AE_n	0	0.134	0.027	0.107	0.13	0.018	0	0.174	0.314

7. Conclusion

The directional distance function allows considerable flexibility in choosing the direction for projecting an inefficient input-output bundle on to the production frontier. However, the analyst must choose a preferred direction irrespective of the data. By contrast, the overall efficiency measure of Ray (2007) yields an optimal direction of projection endogenously. Even though the maximum profit that can be earned at any given pair of input and output price vectors is unique, the different technically efficient projections lead to different amounts of (technically efficient) profit. Also, profit efficiency can be measured either as a ratio or as difference between the actual and the maximum profit of a firm.

Ideally, a ratio measure is preferable because it yields a measure of efficiency that lies between 0 and 1 so long as the actual profit is positive. However, in a multiplicative decomposition when the technical efficiency factor is measured by the ratio of the actual profit over the technically efficient profit, it is price dependent except in the case of the McFadden gauge function. One limitation of the gauge function, however, is that as we have shown, in some cases it might find a point already on the frontier to be technically inefficient. At the same time, an advantage of the Nerlovian efficiency measure is that it will always be positive. Also, the technical efficiency component will be independent of prices. However, in extreme cases, it may exceed 1. Also, using the sum of revenue and cost for normalization is not quite intuitively appealing.

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